Chapter 9

Integration on Manifolds

9.1 Integration in $\mathbb{R}^n$

As we said in Section 8.1, one of the *raison d'être* for differential forms is that they are the objects that can be integrated on manifolds. We will be integrating differential forms that are at least continuous (in most cases, smooth) and with compact support. In the case of forms, ω, on $\mathbb{R}^n$, this means that the closure of the set, $\{x \in \mathbb{R}^n \mid \omega_x \neq 0\}$, is compact. Similarly, for a form, $\omega \in \mathcal{A}^*(M)$, where M is a manifod, the support, $\mathrm{supp}_M(\omega)$, of ω is the closure of the set, $\{p \in M \mid \omega_p \neq 0\}$. We let $\mathcal{A}_c^*(M)$ denote the set of differential forms with compact support on M. If M is a smooth manifold of dimension n, our ultimate goal is to define a linear operator,

$$\int_M : \mathcal{A}_c^n(M) \longrightarrow \mathbb{R},$$

which generalizes in a natural way the usual integral on $\mathbb{R}^n$.

In this section, we assume that $M = \mathbb{R}^n$, or some open subset of $\mathbb{R}^n$. Now, every n-form (with compact support) on $\mathbb{R}^n$ is given by

$$\omega_x = f(x)\, dx_1 \wedge \cdots \wedge dx_n,$$

where f is a smooth function with compact support. Thus, we set

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f(x) dx_1 \cdots dx_n,$$

where the expression on the right-hand side is the usual Riemann integral of f on $\mathbb{R}^n$. Actually, we will need to integrate smooth forms, $\omega \in \mathcal{A}_c^n(U)$, with compact support defined on some open subset, $U \subseteq \mathbb{R}^n$ (with $\mathrm{supp}(\omega) \subseteq U$). However, this is no problem since we still have

$$\omega_x = f(x)\, dx_1 \wedge \cdots \wedge dx_n,$$

where $f \colon U \to \mathbb{R}$ is a smooth function with compact support contained in U and f can be smoothly extended to $\mathbb{R}^n$ by setting it to 0 on $\mathbb{R}^n - \mathrm{supp}(f)$. We write $\int_V \omega$ for this integral.

It is crucial for the generalization of the integral to manifolds to see what the change of variable formula looks like in terms of differential forms.

Proposition 9.1 *Let $\varphi\colon U \to V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. If the Jacobian determinant, $J(\varphi)(x)$, has a constant sign, $\delta = \pm 1$ on U, then for every $\omega \in \mathcal{A}^n_c(V)$, we have*

$$\int_U \varphi^*\omega = \delta \int_V \omega.$$

Proof. We know that ω can be written as

$$\omega_x = f(x)\, dx_1 \wedge \cdots \wedge dx_n, \qquad x \in V,$$

where $f\colon V \to \mathbb{R}$ has compact support. From the example before Proposition 8.6, we have

$$\begin{aligned} (\varphi^*\omega)_y &= f(\varphi(y)) J(\varphi)_y\, dy_1 \wedge \cdots \wedge dy_n \\ &= \delta f(\varphi(y)) |J(\varphi)_y|\, dy_1 \wedge \cdots \wedge dy_n. \end{aligned}$$

On the other hand, the change of variable formula (using φ) is

$$\int_{\varphi(U)} f(x)\, dx_1 \cdots dx_n = \int_U f(\varphi(y))\, |J(\varphi)_y|\, dy_1 \cdots dy_n,$$

so the formula follows. $\square$

We will promote the integral on open subsets of $\mathbb{R}^n$ to manifolds using partitions of unity.

9.2 Integration on Manifolds

Intuitively, for any n-form, $\omega \in \mathcal{A}^n_c(M)$, on a smooth n-dimensional oriented manifold, M, the integral, $\int_M \omega$, is computed by patching together the integrals on small-enough open subsets covering M using a partition of unity. If (U, φ) is a chart such that $\mathrm{supp}(\omega) \subseteq U$, then the form $(\varphi^{-1})^*\omega$ is an n-form on $\mathbb{R}^n$ and the integral, $\int_{\varphi(U)} (\varphi^{-1})^*\omega$, makes sense. The orientability of M is needed to ensure that the above integrals have a consistent value on overlapping charts.

Remark: It is also possible to define integration on non-orientable manifolds using densities but we have no need for this extra generality.

Proposition 9.2 *Let M be a smooth oriented manifold of dimension n. Then, there exists a unique linear operator,*

$$\int_M \colon \mathcal{A}^n_c(M) \longrightarrow \mathbb{R},$$

with the following property: For any $\omega \in \mathcal{A}^n_c(M)$, if $\mathrm{supp}(\omega) \subseteq U$, where (U, φ) is a positively oriented chart, then

$$\int_M \omega = \int_{\varphi(U)} (\varphi^{-1})^*\omega. \tag{$\dagger$}$$

Proof. First, assume that $\mathrm{supp}(\omega) \subseteq U$, where (U, φ) is a positively oriented chart. Then, we wish to set

$$\int_M \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega.$$

However, we need to prove that the above expression does not depend on the choice of the chart. Let (V, ψ) be another chart such that $\mathrm{supp}(\omega) \subseteq V$. The map, $\theta = \psi \circ \varphi^{-1}$, is a diffeomorphism from $W = \varphi(U \cap V)$ to $W' = \psi(U \cap V)$ and, by hypothesis, its Jacobian determinant is positive on W. Since

$$\mathrm{supp}_{\varphi(U)}((\varphi^{-1})^*\omega) \subseteq W, \qquad \mathrm{supp}_{\psi(V)}((\psi^{-1})^*\omega) \subseteq W',$$

and $\theta^* \circ (\psi^{-1})^*\omega = (\varphi^{-1})^* \circ \psi^* \circ (\psi^{-1})^*\omega = (\varphi^{-1})^*\omega$, Proposition 9.1 yields

$$\int_{\varphi(U)} (\varphi^{-1})^*\omega = \int_{\psi(V)} (\psi^{-1})^*\omega,$$

as claimed.

In the general case, using Theorem 3.26, for every open cover of M by positively oriented charts, (U_i, φ_i), we have a partition of unity, $(\rho_i)_{i \in I}$, subordinate to this cover. Recall that

$$\mathrm{supp}(\rho_i) \subseteq U_i, \qquad i \in I.$$

Thus, $\rho_i\omega$ is an n-form whose support is a subset of U_i. Furthermore, as $\sum_i \rho_i = 1$,

$$\omega = \sum_i \rho_i \omega.$$

Define

$$I(\omega) = \sum_i \int_{U_i} \rho_i \omega,$$

where each term in the sum is defined by

$$\int_{U_i} \rho_i\omega = \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* \rho_i \omega,$$

where (U_i, φ_i) is the chart associated with $i \in I$. It remains to show that $I(\omega)$ does not depend on the choice of open cover and on the choice of partition of unity. Let (V_j, ψ_j) be another open cover by positively oriented charts and let $(\theta_j)_{j \in J}$ be a partition of unity subordinate to the open cover, (V_j). Note that

$$\int_{U_i} \rho_i\theta_j\omega = \int_{V_j} \rho_i\theta_j\omega,$$

since $\mathrm{supp}(\rho_i\theta_j\omega) \subseteq U_i \cap V_j$, and as $\sum_i \rho_i = 1$ and $\sum_j \theta_j = 1$, we have

$$\sum_i \int_{U_i} \rho_i\omega = \sum_{i,j} \int_{U_i} \rho_i\theta_j\omega = \sum_{i,j} \int_{V_j} \rho_i\theta_j\omega = \sum_j \int_{V_j} \theta_j\omega,$$

proving that $I(\omega)$ is indeed independent of the open cover and of the partition of unity. The uniqueness assertion is easily proved using a partition of unity. $\square$

The integral of Definition 9.2 has the following properties:

Proposition 9.3 *Let M be an oriented manifold of dimension n. The following properties hold:*

(1) If M is connected, then for every n-form, $\omega \in \mathcal{A}_c^n(M)$, the sign of $\int_M \omega$ changes when the orientation of M is reversed.

(2) For every n-form, $\omega \in \mathcal{A}_c^n(M)$, if $\mathrm{supp}(\omega) \subseteq W$, for some open subset, W, of M, then

$$\int_M \omega = \int_W \omega,$$

where W is given the orientation induced by M.

(3) If $\varphi\colon M \to N$ is an orientation-preserving diffeomorphism, then for every $\omega \in \mathcal{A}_c^n(N)$, we have

$$\int_N \omega = \int_M \varphi^* \omega.$$

Proof. Use a partition of unity to reduce to the case where $\mathrm{supp}(\omega)$ is contained in the domain of a chart and then use Proposition 9.1 and (†) from Proposition 9.2. $\square$

The theory or integration developed so far deals with domains that are not general enough. Indeed, for many applications, we need to integrate over domains with boundaries.

9.3 Integration on Regular Domains and Stokes' Theorem

Given a manifold, M, we define a class of subsets with boundaries that can be integrated on and for which Stokes' Theorem holds. In Warner [147] (Chapter 4), such subsets are called *regular domains* and in Madsen and Tornehave [100] (Chapter 10) they are called *domains with smooth boundary*.

Definition 9.1 Let M be a smooth manifold of dimension n. A subset, $N \subseteq M$, is called a *domain with smooth boundary* (or *codimension zero submanifold with boundary*) iff for every $p \in M$, there is a chart, (U, φ), with $p \in U$, such that

$$\varphi(U \cap N) = \varphi(U) \cap \mathbb{H}^n, \qquad (*)$$

where $\mathbb{H}^n$ is the closed upper-half space,

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$

Note that $(*)$ is automatically satisfied when p is an interior or an exterior point of N, since we can pick a chart such that $\varphi(U)$ is contained in an open half space of $\mathbb{R}^n$ defined by either $x_n > 0$ or $x_n < 0$. If p is a boundary point of N, then $\varphi(p)$ has its last coordinate equal to 0. If M is orientable, then any orientation of M induces an orientation of ∂N, the boundary of N. This follows from the following proposition:

Proposition 9.4 *Let $\varphi\colon \mathbb{H}^n \to \mathbb{H}^n$ be a diffeomorphism with everywhere positive Jacobian determinant. Then, φ induces a diffeomorphism, $\Phi\colon \partial\mathbb{H}^n \to \partial\mathbb{H}^n$, which, viewed as a diffeomorphism of $\mathbb{R}^{n-1}$ also has everywhere positive Jacobian determinant.*

Proof. By the inverse function theorem, every interior point of $\mathbb{H}^n$ is the image of an interior point, so φ maps the boundary to itself. If $\varphi = (\varphi_1, \ldots, \varphi_n)$, then

$$\Phi = (\varphi_1(x_1, \ldots, x_{n-1}, 0), \ldots, \varphi_{n-1}(x_1, \ldots, x_{n-1}, 0)),$$

since $\varphi_n(x_1, \ldots, x_{n-1}, 0) = 0$. It follows that $\frac{\partial \varphi_n}{\partial x_i}(x_1, \ldots, x_{n-1}, 0) = 0$, for $i = 1, \ldots, n-1$, and as φ maps $\mathbb{H}^n$ to itself,

$$\frac{\partial \varphi_n}{\partial x_n}(x_1, \ldots, x_{n-1}, 0) > 0.$$

Now, the Jacobian matrix of φ at $q = \varphi(p) \in \partial\mathbb{H}^n$ is of the form

$$d\varphi_q = \begin{pmatrix} & & & * \\ & d\Phi_q & & \vdots \\ & & & * \\ 0 & \cdots & 0 & \frac{\partial \varphi_n}{\partial x_n}(q) \end{pmatrix}$$

and since $\frac{\partial \varphi_n}{\partial x_n}(q) > 0$ and by hypothesis $\det(d\varphi_q) > 0$, we have $\det(d\Phi_q) > 0$, as claimed. $\square$

In order to make Stokes' formula sign free, if $\mathbb{H}^n$ has the orientation given by $dx_1 \wedge \cdots \wedge dx_n$, then $\mathbb{H}^n$ is given the orientation given by $(-1)^n dx_1 \wedge \cdots \wedge dx_{n-1}$ if $n \geq 2$ and -1 for $n = 1$. This choice of orientation can be explained in terms of the notion of outward directed tangent vector.

Definition 9.2 Given any domain with smooth boundary, $N \subseteq M$, a tangent vector, $w \in T_pM$, at a boundary point, $p \in \partial N$, is *outward directed* iff there is a chart, (U, φ), with $p \in U$ and $\varphi(U \cap N) = \varphi(U) \cap \mathbb{H}^n$ and such that $d\varphi_p(w)$ has a negative n^{th} coordinate $pr_n(d\varphi_p(w))$.

Let (V, ψ) be another chart with $p \in V$. Then, the transition map,

$$\theta = \psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$$

induces a map

$$\varphi(U \cap V) \cap \mathbb{H}^n \longrightarrow \psi(U \cap V) \cap \mathbb{H}^n$$

which restricts to a diffeomorphism

$$\Theta\colon \varphi(U \cap V) \cap \partial \mathbb{H}^n \to \psi(U \cap V) \cap \partial \mathbb{H}^n.$$

The proof of Proposition 9.4 shows that the Jacobian matrix of $d\theta_q$ at $q = \varphi(p) \in \partial \mathbb{H}^n$ is of the form

$$d\theta_q = \begin{pmatrix} & & & * \\ & d\Theta_q & & \vdots \\ & & & * \\ 0 & \cdots & 0 & \frac{\partial \theta_n}{\partial x_n}(q) \end{pmatrix}$$

with $\theta = (\theta_1, \ldots, \theta_n)$ and that $\frac{\partial \theta_n}{\partial x_n}(q) > 0$. As $d\psi_p = d(\psi \circ \varphi^{-1})_q \circ d\varphi_p$, we see that for any $w \in T_pM$ with $pr_n(d\varphi_p(w)) < 0$, we also have $pr_n(d\psi_p(w)) < 0$. Therefore, the negativity condition of Definition does not depend on the chart at p. The following proposition is then easy to show:

Proposition 9.5 *Let $N \subseteq M$ be a domain with smooth boundary where M is a smooth manifold of dimension n.*

(1) The boundary, ∂N, of N is a smooth manifold of dimension $n - 1$.

(2) Assume M is oriented. If $n \geq 2$, there is an induced orientation on ∂N determined as follows: For every $p \in \partial N$, if $v_1 \in T_pM$ is an outward directed tangent vector then a basis, $(v_2, \ldots, v_n)$ for $T_p\partial N$ is positively oriented iff the basis $(v_1, v_2, \ldots, v_n)$ for T_pM is positively oriented. When $n = 1$, every $p \in \partial N$ has the orientation $+1$ iff for every outward directed tangent vector, $v_1 \in T_pM$, the vector v_1 is a positively oriented basis of T_pM.

If M is oriented, then for every n-form, $\omega \in \mathcal{A}^n_c(M)$, the integral $\int_N \omega$ is well-defined. More precisely, Proposition 9.2 can be generalized to domains with a smooth boundary. This can be shown in various ways. In Warner, this is shown by covering N with special kinds of open subsets arising from regular simplices (see Warner [147], Chapter 4). In Madsen and Tornehave [100], it is argued that integration theory goes through for continuous n-forms with compact support. If σ is a volume form on M, then for every continuous function with compact support, f, the map

$$f \mapsto I_\sigma(f) = \int_M f\sigma$$

is a linear positive operator. By Riesz' representation theorem, I_σ determines a positive Borel measure, μ_σ, which satisfies

$$\int_M f d\mu_\sigma = \int_M f\sigma.$$

Then, we can set

$$\int_N \omega = \int_M 1_N \omega,$$

where 1_N is the function with value 1 on N and 0 outside N.

Another way to proceed is to prove an extension of Proposition 9.1 using a slight generalization of the change of variable formula:

Proposition 9.6 *Let $\varphi\colon U \to V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$ and assume that φ maps $U \cap \mathbb{H}^n$ to $V \cap \mathbb{H}^n$. Then, for every smooth function, $f\colon V \to \mathbb{R}$, with compact support,*

$$\int_{V\cap\mathbb{H}^n} f(x)dx_1\cdots dx_n = \int_{U\cap\mathbb{H}^n} f(\varphi(y))\,|J(\varphi)_y|\,dy_1\cdots dy_n.$$

We now have all the ingredient to state Stokes's formula. We omit the proof as it can be found in many places (for example, Warner [147], Chapter 4, Bott and Tu [19], Chapter 1, and Madsen and Tornehave [100], Chapter 10). The proof is fairly easy and it is not particularly illuminating, although one has to be very careful about matters of orientation.

Theorem 9.7 *(Stokes' Theorem) Let $N \subseteq M$ be a domain with smooth boundary where M is a smooth oriented manifold of dimension n, give ∂N the orientation induced by M and let $i\colon \partial N \to M$ be the inclusion map. For every differential form with compact support, $\omega \in \mathcal{A}_c^{n-1}(M)$, we have*

$$\int_{\partial N} i^*\omega = \int_N d\omega.$$

In particular, if $N = M$ is a smooth oriented manifold with boundary, then

$$\int_{\partial M} i^*\omega = \int_M d\omega$$

and if M is a smooth oriented manifold without boundary, then

$$\int_M d\omega = 0.$$

Of course, $i^*\omega$ is the restriction of ω to ∂N and for simplicity of notation, $i^*\omega$ is usually written ω and Stokes' formula is written

$$\int_{\partial N} \omega = \int_N d\omega.$$

9.4 Integration on Riemannian Manifolds and Lie Groups

We saw in Section 8.6 that every orientable Riemannian manifold has a uniquely defined volume form, Vol_M (see Proposition 8.26). given any smooth function, f, with compact support on M, we define the *integral of f over M* by

$$\int_M f = \int_M f\,\mathrm{Vol}_M.$$

Actually, it is possible to define the integral, $\int_M f$, even if M is not orientable but we do not need this extra generality. If M is compact, then $\int_M 1_M = \int_M \mathrm{Vol}_M$ is the *volume* of M (where 1_M is the constant function with value 1).

If M and N are Riemannian manifolds, then we have the following version of Proposition 9.3 (3):

Proposition 9.8 *If M and N are oriented Riemannian manifolds and if $\varphi\colon M \to N$ is an orientation preserving diffeomorphism, then for every function, $f \in C^\infty(M)$, with compact support, we have*

$$\int_N f\,\mathrm{Vol}_N = \int_M f \circ \varphi\,|\det(d\varphi)|\,\mathrm{Vol}_M,$$

where $f\circ\varphi\,|\det(d\varphi)|$ denotes the function, $p \mapsto f(\varphi(p))|\det(d\varphi_p)|$, with $d\varphi_p\colon T_pM \to T_{\varphi(p)}N$. In particular, if φ is an orientation preserving isometry (see Definition 7.11), then

$$\int_N f\,\mathrm{Vol}_N = \int_M f \circ \varphi\,\mathrm{Vol}_M.$$

We often denote $\int_M f\,\mathrm{Vol}_M$ by $\int_M f(t)dt$.

If G is a Lie group, we know from Section 8.6 that G is always orientable and that G possesses left-invariant volume forms. Since $\dim(\bigwedge^n \mathfrak{g}^*) = 1$ if $\dim(G) = n$ and since every left-invariant volume form is determined by its value at the identity, the space of left-invariant volume forms on G has dimension 1. If we pick some left-invariant volume form, ω, defining the orientation of G, then every other left-invariant volume form is proportional to ω. Given any smooth function, f, with compact support on G, we define the *integral of f over G (w.r.t. ω)* by

$$\int_G f = \int_G f\omega.$$

This integral depends on ω but since ω is defined up to some positive constant, so is the integral. When G is compact, we usually pick ω so that

$$\int_G \omega = 1.$$

For every $g \in G$, as ω is left-invariant, $L_g^*\omega = \omega$, so L_g^* is an orientation-preserving diffeomorphism and by Proposition 9.3 (3),

$$\int_G f\omega = \int_G L_g^*(f\omega),$$

so we get

$$\int_G f = \int_G f\omega = \int_G L_g^*(f\omega) = \int_G L_g^*f\,L_g^*\omega = \int_G (f \circ L_g)\omega = \int_G f \circ L_g.$$

The property

$$\int_G f = \int_G f \circ L_g$$

is called *left-invariance*.

It is then natural to ask when our integral is right-invariant, that is, when

$$\int_G f = \int_G f \circ R_g.$$

Observe that $R_g^*\omega$ is left-invariant, since

$$L_h^* R_g^* \omega = R_g^* L_h^* \omega = R_g^* \omega.$$

It follows that $R_g^*\omega$ is some constant multiple of ω, and so, there is a function, $\overline{\Delta}\colon G \to \mathbb{R}$ such that

$$R_g^*\omega = \overline{\Delta}(g)\omega.$$

One can check that $\overline{\Delta}$ is smooth and we let

$$\Delta(g) = |\overline{\Delta}(g)|.$$

Clearly,

$$\Delta(gh) = \Delta(g)\Delta(h),$$

so Δ is a homorphism of G into $\mathbb{R}_+$. The function Δ is called the *modular function of* G. Now, by Proposition 9.3 (3), as R_g^* is an orientation-preserving diffeomorphism,

$$\int_G f\omega = \int_G R_g^*(f\omega) = \int_G R_g^* f \circ R_g^* \omega = \int_G (f \circ R_g)\Delta(g)\omega$$

or, equivalently,

$$\int_G f\omega = \Delta(g^{-1}) \int_G (f \circ R_g)\omega.$$

It follows that if ω_l is any left-invariant volume form on G and if ω_r is any right-invariant volume form in G, then

$$\omega_r(g) = c\Delta(g^{-1})\omega_l(g),$$

for some constant $c \neq 0$. Indeed, if let $\omega(g) = \overline{\Delta}(g^{-1})\omega_l(g)$, then

$$\begin{aligned} R_h^*\omega &= \overline{\Delta}((gh)^{-1}) R_h^* \omega_l \\ &= \overline{\Delta}(h)^{-1} \overline{\Delta}(g^{-1}) \overline{\Delta}(h) \omega_l \\ &= \overline{\Delta}(g^{-1})\omega_l, \end{aligned}$$

which shows that ω is right-invariant and thus, $\omega_r(g) = c\Delta(g^{-1})\omega_l(g)$, as claimed (since $\overline{\Delta}(g^{-1}) = \pm\Delta(g^{-1})$). Actually, it is not difficult to prove that

$$\Delta(g) = |\det(\mathrm{Ad}(g^{-1}))|.$$

For this, recall that $\mathrm{Ad}(g) = d(L_g \circ R_{g^{-1}})_1$. For any left-invariant n-form, $\omega \in \bigwedge^n \mathfrak{g}^*$, we claim that

$$(R_g^*\omega)_h = \det(\mathrm{Ad}(g^{-1}))\omega_h,$$

which shows that $\Delta(g) = |\det(\mathrm{Ad}(g^{-1}))|$. Indeed, for all $v_1, \ldots, v_n \in T_hG$, we have

$$\begin{aligned}
&(R_g^*\omega)_h(v_1, \ldots, v_n)\\
&\quad = \omega_{hg}(d(R_g)_h(v_1), \ldots, d(R_g)_h(v_n))\\
&\quad = \omega_{hg}(d(L_g \circ L_{g^{-1}} \circ R_g \circ L_h \circ L_{h^{-1}})_h(v_1), \ldots, d(L_g \circ L_{g^{-1}} \circ R_g \circ L_h \circ L_{h^{-1}})_h(v_n))\\
&\quad = \omega_{hg}(d(L_h \circ L_g \circ L_{g^{-1}} \circ R_g \circ L_{h^{-1}})_h(v_1), \ldots, d(L_h \circ L_g \circ L_{g^{-1}} \circ R_g \circ L_{h^{-1}})_h(v_n))\\
&\quad = \omega_{hg}(d(L_{hg} \circ L_{g^{-1}} \circ R_g \circ L_{h^{-1}})_h(v_1), \ldots, d(L_{hg} \circ L_{g^{-1}} \circ R_g \circ L_{h^{-1}})_h(v_n))\\
&\quad = \omega_{hg}\big(d(L_{hg})_1(\mathrm{Ad}(g^{-1})(d(L_{h^{-1}})_h(v_1))), \ldots, d(L_{hg})_1(\mathrm{Ad}(g^{-1})(d(L_{h^{-1}})_h(v_n)))\big)\\
&\quad = (L_{hg}^*\omega)_1\big(\mathrm{Ad}(g^{-1})(d(L_{h^{-1}})_h(v_1)), \ldots, \mathrm{Ad}(g^{-1})(d(L_{h^{-1}})_h(v_n))\big)\\
&\quad = \omega_1\big(\mathrm{Ad}(g^{-1})(d(L_{h^{-1}})_h(v_1)), \ldots, \mathrm{Ad}(g^{-1})(d(L_{h^{-1}})_h(v_n))\big)\\
&\quad = \det(\mathrm{Ad}(g^{-1}))\, \omega_1\big(d(L_{h^{-1}})_h(v_1), \ldots, d(L_{h^{-1}})_h(v_n)\big)\\
&\quad = \det(\mathrm{Ad}(g^{-1}))\, (L_{h^{-1}}^*\omega)_h(v_1, \ldots, v_n)\\
&\quad = \det(\mathrm{Ad}(g^{-1}))\, \omega_h(v_1, \ldots, v_n),
\end{aligned}$$

where we used the left-invariance of ω twice.

Consequently, our integral is right-invariant iff $\Delta \equiv 1$ on G. Thus, our integral is not always right-invariant. When it is, *i.e.* when $\Delta \equiv 1$ on G, we say that G *is unimodular*. This happens in particular when G is compact, since in this case,

$$1 = \int_G \omega = \int_G 1_G\omega = \int_G \Delta(g)\omega = \Delta(g) \int_G \omega = \Delta(g),$$

for all $g \in G$. Therefore, for a *compact Lie group, G*, our integral is both left and right invariant. We say that our integral is *bi-invariant*.

As a matter of notation, the integral $\int_G f = \int_G f\omega$ is often written $\int_G f(g)dg$. Then, left-invariance can be expressed as

$$\int_G f(g)dg = \int_G f(hg)dg$$

and right-invariance as

$$\int_G f(g)dg = \int_G f(gh)dg,$$

for all $h \in G$. If ω is left-invariant, then it can be shown that

$$\int_G f(g^{-1})\Delta(g^{-1})dg = \int_G f(g)dg.$$

Consequently, if G is unimodular, then

$$\int_G f(g^{-1})dg = \int_G f(g)dg.$$

In general, if G is not unimodular, then $\omega_l \neq \omega_r$. A simple example is the group, G, of affine transformations of the real line, which can be viewed as the group of matrices of the form

$$A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \qquad a, b, \in \mathbb{R},\ a \neq 0.$$

Then, it it is easy to see that the left-invariant volume form and the right-invariant volume form on G are given by

$$\omega_l = \frac{dadb}{a^2}, \qquad \omega_r = \frac{dadb}{a},$$

and so, $\Delta(A) = |a^{-1}|$.

Remark: By the Riesz' representation theorem, ω defines a positive measure, μ_ω, which satisfies

$$\int_G f d\mu_\omega = \int_G f\omega.$$

Using what we have shown, this measure is left-invariant. Such measures are called *left Haar measures* and similarly, we have *right Haar measures*. It can be shown that every two left Haar measures on a Lie group are proportional (see Knapp, [89], Chapter VIII). Given a left Haar measure, μ, the function, Δ, such that

$$\mu(R_g h) = \Delta(g)\mu(h)$$

for all $g, h \in G$ is the *modular function* of G. However, beware that some authors, including Knapp, use $\Delta(g^{-1})$ instead of $\Delta(g)$. As above, we have

$$\Delta(g) = |\det(\mathrm{Ad}(g^{-1}))|.$$

Beware that authors who use $\Delta(g^{-1})$ instead of $\Delta(g)$, give a formula where $\mathrm{Ad}(g)$ appears instead of $\mathrm{Ad}(g^{-1})$. Again, G is *unimodular* iff $\Delta \equiv 1$. It can be shown that compact, semisimple, reductive and nilpotent Lie groups are unimodular (for instance, see Knapp, [89], Chapter VIII). On such groups, left Haar measures are also right Haar measures (and vice versa). In this case, we can speak of *Haar measures* on G. For more details on Haar measures on locally compact groups and Lie groups, we refer the reader to Folland [54] (Chapter 2), Helgason [72] (Chapter 1) and Dieudonné [47] (Chapter XIV).

Chapter 10

Distributions and the Frobenius Theorem

10.1 Tangential Distributions, Involutive Distributions

Given any smooth manifold, M, (of dimension n) for any smooth vector field, X, on M, we know from Section 3.5 that for every point, $p \in M$, there is a unique maximal integral curve through p. Furthermore, any two distinct integral curves do not intersect each other and the union of all the integral curves is M itself. A nonvanishing vector field, X, can be viewed as the smooth assignment of a one-dimensional vector space to every point of M, namely, $p \mapsto \mathbb{R}X_p \subseteq T_pM$, where $\mathbb{R}X_p$ denotes the line spanned by X_p. Thus, it is natural to consider the more general situation where we fix some integer, r, with $1 \leq r \leq n$ and we have an assignment, $p \mapsto D(p) \subseteq T_pM$, where $D(p)$ is some r-dimensional subspace of T_pM such that $D(p)$ "varies smoothly" with $p \in M$. Is there a notion of integral manifold for such assignments? Do they always exist?

It is indeed possible to generalize the notion of integral curve and to define integral manifolds but, unlike the situation for vector fields ($r = 1$), not every assignment, D, as above, possess an integral manifold. However, there is a necessary and sufficient condition for the existence of integral manifolds given by the Frobenius Theorem. This theorem has several equivalent formulations. First, we will present a formulation in terms of vector fields. Then, we will show that there are advantages in reformulating the notion of involutivity in terms of differential ideals and we will state a differential form version of the Frobenius Theorem. The above versions of the Frobenius Theorem are "local". We will briefly discuss the notion of foliation and state a global version of the Frobenius Theorem.

Since Frobenius' Theorem is a standard result of differential geometry, we will omit most proofs and instead refer the reader to the literature. A complete treatment of Frobenius' Theorem can be found in Warner [147], Morita [114] and Lee [98].

Our first task is to define precisely what we mean by a smooth assignment, $p \mapsto D(p) \subseteq T_pM$, where $D(p)$ is an r-dimensional subspace.

Definition 10.1 Let M be a smooth manifold of dimension n. For any integer r, with $1 \leq r \leq n$, an r-dimensional *tangential distribution* (for short, a *distribution*) is a map, $D\colon M \to TM$, such that

(a) $D(p) \subseteq T_pM$ is an r-dimensional subspace for all $p \in M$.

(b) For every $p \in M$, there is some open subset, U, with $p \in U$, and r smooth vector fields, $X_1, \ldots, X_r$, defined on U, such that $(X_1(q), \ldots, X_r(q))$ is a basis of $D(q)$ for all $q \in U$. We say that D is *locally spanned* by $X_1, \ldots, X_r$.

An immersed submanifold, N, of M is an *integral manifold* of D iff $D(p) = T_pN$, for all $p \in N$. We say that D is *completely integrable* iff there exists an integral manifold of D through every point of M.

We also write D_p for $D(p)$.

Remarks:

(1) An r-dimensional distribution, D, is just a smooth subbundle of TM.

(2) An integral manifold is only an immersed submanifold, not necessarily an embedded submanifold.

(3) Some authors (such as Lee) reserve the locution "completely integrable" to a seemingly strongly condition (See Lee [98], Chapter 19, page 500). This condition is in fact equivalent to "our" definition (which seems the most commonly adopted).

(4) Morita [114] uses a stronger notion of integral manifold, namely, an integral manifold is actually an embedded manifold. Most of the results, including Frobenius Theorem still hold but maximal integral manifolds are immersed but not embedded manifolds and this is why most authors prefer to use the weaker definition (immersed manifolds).

Here is an example of a distribution which does not have any integral manifolds: This is the two-dimensional distribution in $\mathbb{R}^3$ spanned by the vector fields

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \qquad X = \frac{\partial}{\partial y}.$$

We leave it as an exercise to the reader to show that the above distribution is not integrable.

The key to integrability is an involutivity condition. Here is the definition.

Definition 10.2 Let M be a smooth manifold of dimension n and let D be an r-dimensional distribution on M. For any smooth vector field, X, we say that X *belongs to* D (or *lies in* D) iff $X_p \in D_p$, for all $p \in M$. We say that D is *involutive* iff for any two smooth vector fields, X, Y, on M, if X and Y belong to D, then $[X, Y]$ also belongs to D.

Proposition 10.1 *Let M be a smooth manifold of dimension n. If an r-dimensional distribution, D, is completely integrable, then D is involutive.*

Proof. A proof can be found in in Warner [147] (Chapter 1), and Lee [98] (Proposition 19.3). These proofs use Proposition 3.14. Another proof is given in Morita [114] (Section 2.3) but beware that Morita defines an integral manifold to be an embedded manifold. $\square$

In the example before Definition 10.1, we have

$$[X, Y] = -\frac{\partial}{\partial z},$$

so this distribution is not involutive. Therefore, by Proposition 10.1, this distribution is not completely integrable.

10.2 Frobenius Theorem

Frobenius' Theorem asserts that the converse of Proposition 10.1 holds. Although we do not intend to prove it in full, we would like to explain the main idea of the proof of Frobenius' Theorem. It turns out that the involutivity condition of two vector fields is equivalent to the commutativity of their corresponding flows and this is the crucial fact used in the proof.

Given a manifold, M, we sa that two vector fields, X and Y are *mutually commutative* iff $[X, Y] = 0$. For example, on $\mathbb{R}^2$, the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are commutative but $\frac{\partial}{\partial x}$ and $x\frac{\partial}{\partial y}$ are not.

Recall from Definition 3.23 that we denote by Φ^X the (global) flow of the vector field, X. For every $p \in M$, the map, $t \mapsto \Phi^X(t, p) = \gamma_p(t)$ is the maximal integral curve through p. We also write $\Phi_t(p)$ for $\Phi^X(t, p)$ (dropping X). Recall that the map, $p \mapsto \Phi_t(p)$, is a diffeomorphism on its domain (an open subset of M). For the next proposition, given two vector fields, X and Y, we will write Φ for the flow associated with X and Ψ for the flow associated with Y.

Proposition 10.2 *Given a manifold, M, for any two smooth vector fields, X and Y, the following conditions are equivalent:*

(1) X and Y are mutually commutative (i.e. $[X, Y] = 0$).

(2) Y is invariant under Φ_t, that is, $(\Phi_t)_*Y = Y$, whenever the left-hand side is defined.

(3) X is invariant under Ψ_s, that is, $(\Psi_s)_*X = X$, whenever the left-hand side is defined.

(4) The maps Φ_t and Ψ_t are mutually commutative. This means that

$$\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t,$$

for all s, t such that both sides are defined.

(5) $\mathcal{L}_X Y = 0$.

(6) $\mathcal{L}_Y X = 0$.

(In (5) $\mathcal{L}_X Y$ *is the Lie derivative and similarly in (6).)*

Proof. A proof can be found in Lee [98] (Chapter 18, Proposition 18.5) and in Morita [114] (Chapter 2, Proposition 2.18). For example, to prove the implication (2) $\Longrightarrow$ (4), we observe that if φ is a diffeomorphism on some open subset, U, of M, then the integral curves of $\varphi_* Y$ through a point $p \in M$ are of the form $\varphi \circ \gamma$, where γ is the integral curve of Y through $\varphi^{-1}(p)$. Consequently, the local one-parameter group generated by $\varphi_* Y$ is $\{\varphi \circ \Psi_s \circ \varphi^{-1}\}$. If we apply this to $\varphi = \Phi_t$, as $(\Phi_t)_* Y = Y$, we get $\Phi_t \circ \Psi_s \circ \Phi_t^{-1} = \Psi_s$ and hence, $\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t$. $\square$

In order to state our first version of the Frobenius Theorem we make the following definition:

Definition 10.3 Let M be a smooth manifold of dimension n. Given any smooth r-dimensional distribution, D, on M, a chart, (U, φ), is *flat for D* iff

$$\varphi(U) \cong U' \times U'' \subseteq \mathbb{R}^r \times \mathbb{R}^{n-r},$$

where U' and U'' are connected open subsets such that for every $p \in U$, the distribution D is spanned by the vector fields

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r}.$$

If (U, φ) is flat for D, then it is clear that each slice of (U, φ),

$$S_c = \{q \in U \mid x_{r+1} = c_{r+1}, \ldots, x_n = c_n\},$$

is an integral manifold of D, where $x_i = pr_i \circ \varphi$ is the i^{th}-coordinate function on U and $c = (c_{r+1}, \ldots, c_n) \in \mathbb{R}^{n-r}$ is a fixed vector.

Theorem 10.3 *(Frobenius) Let M be a smooth manifold of dimension n. A smooth r-dimensional distribution, D, on M is completely integrable iff it is involutive. Furthermore, for every $p \in U$, there is flat chart, (U, φ), for D with $p \in U$, so that every slice of (U, φ) is an integral manifold of D.*

Proof. A proof of Theorem 10.3 can be found in Warner [147] (Chapter 1, Theorem 1.60), Lee [98] (Chapter 19, Theorem 19.10) and Morita [114] (Chapter 2, Theorem 2.17). Since we already have Proposition 10.1, it is only necessary to prove that if a distribution is involutive then it is completely integrable. Here is a sketch of the proof, following Morita.

Pick any $p \in M$. As D is a smooth distribution, we can find some chart, (U, φ), with $p \in U$, and some vector fields, $Y_1, \ldots, Y_r$, so that $Y_1(q), \ldots, Y_r(q)$ are linearly independent and span D_q for all $q \in U$. Locally, we can write

$$Y_i = \sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial x_j}, \qquad i = 1, \ldots, r.$$

Since the Y_i are linearly independent, by renumbering the coordinates if necessary, we may assume that the $r \times r$ matrices

$$A(q) = (a_{ij}(q)) \qquad q \in U$$

are invertible. Then, the inverse matrices, $B(q) = A^{-1}(q)$ define $r \times r$ functions, $b_{ij}(q)$ and let

$$X_i = \sum_{j=1}^{r} b_{ij} Y_j, \qquad j = 1, \ldots, r.$$

Now, in matrix form,

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_r \end{pmatrix} = \begin{pmatrix} A & R \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1}, \\ \vdots \\ \frac{\partial}{\partial x_n}, \end{pmatrix}$$

for some $r \times (n - r)$ matrix, R and

$$\begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix} = B \begin{pmatrix} Y_1 \\ \vdots \\ Y_r \end{pmatrix},$$

so we get

$$\begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix} = \begin{pmatrix} I & BR \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix},$$

that is,

$$X_i = \frac{\partial}{\partial x_i} + \sum_{j=r+1}^{n} c_{ij} \frac{\partial}{\partial x_j}, \qquad i = 1, \ldots, r, \tag{$*$}$$

where the c_{ij} are functions defined on U. Obviously, $X_1, \ldots, X_r$ are linearly independent and they span D_q for all $q \in U$. Since D is involutive, there are some functions, f_k, defined on U, so that

$$[X_i, X_j] = \sum_{k=1}^{r} f_k X_k.$$

On the other hand, by $(*)$, each $[X_i, X_j]$ is a linear combination of $\frac{\partial}{\partial x_{r+1}}, \ldots, \frac{\partial}{\partial x_n}$. Therefore, $f_k = 0$, for $k = 1, \ldots, r$, which shows that

$$[X_i, X_j] = 0, \qquad 1 \leq i, j \leq r,$$

that is, the vector fields $X_1, \ldots, X_r$ are mutually commutative.

Let Φ_t^i be the local one-parameter group associated with X_i. By Proposition 10.2 (4), the Φ_t^i commute, that is,

$$\Phi_t^i \circ \Phi_s^j = \Phi_s^j \circ \Phi_t^i \qquad 1 \leq i, j \leq r,$$

whenever both sides are defined. We can pick a sufficiently open subset, V, in $\mathbb{R}^r$ containing the origin and define the map, $\Phi \colon V \to U$ by

$$\Phi(t_1, \ldots, t_r) = \Phi_{t_1}^1 \circ \cdots \circ \Phi_{t_r}^r(p).$$

Clearly, Φ is smooth and using the fact that each X_i is invariant under each Φ_s^j, for $j \neq i$, and

$$d\Phi_p^i\left(\frac{\partial}{\partial t_i}\right) = X_i(p),$$

we get

$$d\Phi_p\left(\frac{\partial}{\partial t_i}\right) = X_i(p).$$

As $X_1, \ldots, X_r$ are linearly independent, we deduce that $d\Phi_p \colon T_0\mathbb{R}^r \to T_pM$ is an injection and thus, we may assume by shrinking V if necessary that our map, $\Phi \colon V \to M$, is an embedding. But then, $N = \Phi(V)$ is a submanifold of M and it only remains to prove that N is an integral manifold of D through p.

Obviously, $T_pN = D_p$, so we just have to prove that $T_qN = D_qN$ for all $q \in N$. Now, for every $q \in N$, we can write

$$q = \Phi(t_1, \ldots, t_r) = \Phi_{t_1}^1 \circ \cdots \circ \Phi_{t_r}^r(p),$$

for some $(t_1, \ldots, t_r) \in V$. Since the Φ_t^i commute, for any i, with $1 \leq i \leq r$, we can write

$$q = \Phi_{t_i}^i \circ \Phi_{t_1}^1 \circ \cdots \circ \Phi_{t_{i-1}}^{i-1} \circ \Phi_{t_{i+1}}^{i+1} \circ \cdots \circ \Phi_{t_r}^r(p).$$

If we fix all the t_j but t_i and vary t_i by a small amount, we obtain a curve in N through q and this is an orbit of Φ_t^i. Therefore, this curve is an integral curve of X_i through q whose velocity vector at q is equal to $X_i(q)$ and so, $X_i(q) \in T_qN$. Since the above reasoning holds for all i, we get $T_qN = D_q$, as claimed. Therefore, N is an integral manifold of D through p. $\square$

In preparation for a global version of Frobenius Theorem in terms of foliations, we state the following Proposition proved in Lee [98] (Chapter 19, Proposition 19.12):

Proposition 10.4 *Let M be a smooth manifold of dimension n and let D be an involutive r-dimensional distribution on M. For every flat chart, (U, φ), for D, for every integral manifold, N, of D, the set $N \cap U$ is a countable disjoint union of open parallel k-dimensional slices of U, each of which is open in N and embedded in M.*

We now describe an alternative method for describing involutivity in terms of differential forms.

10.3 Differential Ideals and Frobenius Theorem

First, we give a smoothness criterion for distributions in terms of one-forms.

Proposition 10.5 *Let M be a smooth manifold of dimension n and let D be an assignment, $p \mapsto D_p \subseteq T_pM$, of some r-dimensional subspace of T_pM, for all $p \in M$. Then, D is a smooth distribution iff for every $p \in U$, there is some open subset, U, with $p \in U$, and some linearly independent one-forms, $\omega_1, \ldots, \omega_{n-r}$, defined on U, so that*

$$D_q = \{u \in T_qM \mid (\omega_1)_q(u) = \cdots = (\omega_{n-r})_q(u) = 0\}, \qquad \textit{for all } q \in U.$$

Proof. Proposition 10.5 is proved in Lee [98] (Chapter 19, Lemma 19.5). The idea is to either extend a set of linearly independent differential one-forms to a coframe and then consider the dual frame or to extend some linearly independent vector fields to a frame and then take the dual basis. $\square$

Proposition 10.5 suggests the following definition:

Definition 10.4 Let M be a smooth manifold of dimension n and let D be an r-dimensional distibution on M. Some linearly independent one-forms, $\omega_1, \ldots, \omega_{n-r}$, defined some open subset, $U \subseteq M$, are called *local defining one-forms* for D if

$$D_q = \{u \in T_qM \mid (\omega_1)_q(u) = \cdots = (\omega_{n-r})_q(u) = 0\}, \qquad \text{for all } q \in U.$$

We say that a k-form, $\omega \in \mathcal{A}^k(M)$, *annihilates* D iff

$$\omega_q(X_1(q), \ldots, X_r(q)) = 0,$$

for all $q \in M$ and for all vector fields, $X_1, \ldots, X_r$, belonging to D. We write

$$\mathfrak{I}^k(D) = \{\omega \in \mathcal{A}^k(M) \mid \omega_q(X_1(q), \ldots, X_r(q)) = 0\},$$

for all $q \in M$ and for all vector fields, $X_1, \ldots, X_r$, belonging to D and we let

$$\mathfrak{I}(D) = \bigoplus_{k=1}^{n} \mathfrak{I}^k(D).$$

Thus, $\mathfrak{I}(D)$ is the collection of differential forms that "vanish on D." In the classical terminology, a system of local defining one-forms as above is called a *system of Pfaffian equations*.

It turns out that $\mathfrak{I}(D)$ is not only a vector space but also an ideal of $\mathcal{A}^\bullet(M)$.

A subspace, $\mathfrak{I}$, of $\mathcal{A}^\bullet(M)$ is an *ideal* iff for every $\omega \in \mathfrak{I}$, we have $\theta \wedge \omega \in \mathfrak{I}$ for every $\theta \in \mathcal{A}^\bullet(M)$.

Proposition 10.6 *Let M be a smooth n-dimensional manifold and D be an r-dimensional distribution. If $\mathfrak{I}(D)$ is the space of forms annihilating D then the following hold:*

(a) $\mathfrak{I}(D)$ is an ideal in $\mathcal{A}^\bullet(M)$.

(b) $\mathfrak{I}(D)$ is locally generated by $n-r$ linearly independent one-forms, which means: For every $p \in U$, there is some open subset, $U \subseteq M$, with $p \in U$ and a set of linearly independent one-forms, $\omega_1, \ldots, \omega_{n-r}$, defined on U, so that

(i) If $\omega \in \mathfrak{I}^k(D)$, then $\omega \restriction U$ belongs to the ideal in $\mathcal{A}^\bullet(U)$ generated by $\omega_1, \ldots, \omega_{n-r}$, that is,

$$\omega = \sum_{i=1}^{n-r} \theta_i \wedge \omega_i, \qquad \text{on } U,$$

for some $(k-1)$-forms, $\theta_i \in \mathcal{A}^{k-1}(U)$.

(ii) If $\omega \in \mathcal{A}^k(M)$ and if there is an open cover by subsets U (as above) such that for every U in the cover, $\omega \restriction U$ belongs to the ideal generated by $\omega_1, \ldots, \omega_{n-r}$, then $\omega \in \mathfrak{I}(D)$.

(c) If $\mathfrak{I} \subseteq \mathcal{A}^\bullet(M)$ is an ideal locally generated by $n-r$ linearly independent one-forms, then there exists a unique smooth r-dimensional distribution, D, for which $\mathfrak{I} = \mathfrak{I}(D)$.

Proof. Proposition 10.6 is proved in Warner (Chapter 2, Proposition 2.28). See also Morita [114] (Chapter 2, Lemma 2.19) and Lee [98] (Chapter 19, page 498-500). □

In order to characterize involutive distributions, we need the notion of differential ideal.

Definition 10.5 Let M be a smooth manifold of dimension n. An ideal, $\mathfrak{I} \subseteq \mathcal{A}^\bullet(M)$, is a *differential ideal* iff it is closed under exterior differentiation, that is

$$d\omega \in \mathfrak{I} \quad \text{whenever} \quad \omega \in \mathfrak{I},$$

which we also express by $d\mathfrak{I} \subseteq \mathfrak{I}$.

Here is the differential ideal criterion for involutivity.

Proposition 10.7 *Let M be a smooth manifold of dimension n. A smooth r-dimensional distribution, D, is involutive iff the ideal, $\mathfrak{I}(D)$, is a differential ideal.*

Proof. Proposition 10.7 is proved in Warner [147] (Chapter 2, Proposition 2.30), Morita [114] (Chapter 2, Proposition 2.20) and Lee [98] (Chapter 19, Proposition 19.19). Here is one direction of the proof. Assume $\mathfrak{I}(D)$ is a differential ideal. We know that for any one-form, ω,

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]),$$

for any vector fields, X, Y. Now, if $\omega_1, \ldots, \omega_{n-r}$ are linearly independent one-forms that define D locally on U, using a bump function, we can extend $\omega_1, \ldots, \omega_{n-r}$ to M and then using the above equation, for any vector fields X, Y belonging to D, we get

$$\omega_i([X, Y]) = X(\omega_i(Y)) - Y(\omega_i(X)) - d\omega_i(X, Y) = 0,$$

and since $\omega_i(X) = \omega_i(Y) = d\omega_i(X, Y) = 0$, we get $\omega_i([X, Y]) = 0$ for $i = 1, \ldots, n-r$, which means that $[X, Y]$ belongs to D. $\square$

Using Proposition 10.6, we can give a more concrete criterion: D is involutive iff for every local defining one-forms, $\omega_1, \ldots, \omega_{n-r}$, for D (on some open subset, U), there are some one-forms, $\omega_{ij} \in \mathcal{A}^1(U)$, so that

$$d\omega_i = \sum_{j=1}^{n-r} \omega_{ij} \wedge \omega_j \qquad (i = 1, \ldots, n-r).$$

The above conditions are often called the *integrability conditions*.

Definition 10.6 Let M be a smooth manifold of dimension n. Given any ideal $\mathfrak{I} \subseteq \mathcal{A}^\bullet(M)$, an immersed manifold, (M, ψ), of M is an *integral manifold of* $\mathfrak{I}$ iff

$$\psi^*\omega = 0, \qquad \text{for all } \omega \in \mathfrak{I}.$$

A connected integral manifold of the ideal $\mathfrak{I}$ is *maximal* iff its image is not a proper subset of the image of any other connected integral manifold of $\mathfrak{I}$.

Finally, here is the differential form version of the Frobenius Theorem.

Theorem 10.8 *(Frobenius Theorem, Differential Ideal Version) Let M be a smooth manifold of dimension n. If $\mathfrak{I} \subseteq \mathcal{A}^\bullet(M)$ is a differential ideal locally generated by $n-r$ linearly independent one-forms, then for every $p \in M$, there exists a unique maximal, connected, integral manifold of $\mathfrak{I}$ through p and this integral manifold has dimension r.*

Proof. Theorem 10.8 is proved in Warner [147]. This theorem follows immediately from Theorem 1.64 in Warner [147].

Another version of the Frobenius Theorem goes as follows:

Theorem 10.9 *(Frobenius Theorem, Integrability Conditions Version) Let M be a smooth manifold of dimension n. An r-dimensional distribution, D, on M is completely integrable iff for every local defining one-forms, $\omega_1, \ldots, \omega_{n-r}$, for D (on some open subset, U), there are some one-forms, $\omega_{ij} \in \mathcal{A}^1(U)$, so that we have the integrability conditions*

$$d\omega_i = \sum_{j=1}^{n-r} \omega_{ij} \wedge \omega_j \qquad (i = 1, \ldots, n-r).$$

There are applications of Frobenius Theorem (in its various forms) to systems of partial differential equations but we will not deal with this subject. The reader is advised to consult Lee [98], Chapter 19, and the references there.

10.4 A Glimpse at Foliations and a Global Version of Frobenius Theorem

All the maximal integral manifolds of an r-dimensional involutive distribution on a manifold, M, yield a decomposition of M with some nice properties, those of a foliation.

Definition 10.7 Let M be a smooth manifold of dimension n. A family, $\mathcal{F} = \{\mathcal{F}_\alpha\}_\alpha$, of subsets of M is a *k-dimensional foliation* iff it is a family of pairwise disjoint, connected, immersed k-dimensional submanifolds of M, called the *leaves* of the foliation, whose union is M and such that, for every $p \in M$, there is a chart, (U, φ), with $p \in U$, called a *flat chart* for the foliation and the following property holds:

$$\varphi(U) \cong U' \times U'' \subseteq \mathbb{R}^r \times \mathbb{R}^{n-r},$$

where U' and U'' are some connected open subsets and for every leaf, $\mathcal{F}_\alpha$, of the foliation, if $\mathcal{F}_\alpha \cap U \neq \emptyset$, then $\mathcal{F}_\alpha \cap U$ is a countable union of k-dimensional slices given by

$$x_{r+1} = c_{r+1}, \ldots, x_n = c_n,$$

for some constants, $c_{r+1}, \ldots, c_n \in \mathbb{R}$.

The structure of a foliation can be very complicated. For instance, the leaves can be dense in M. For example, there are spirals on a torus that form the leaves of a foliation (see Lee [98], Example 19.9). Foliations are in one-to-one correspondence with involutive distributions.

Proposition 10.10 *Let M be a smooth manifold of dimension n. For any foliation, $\mathcal{F}$, on M, the family of tangent spaces to the leaves of $\mathcal{F}$ forms an involutive distribution on M.*

The converse to the above proposition may be viewed as a global version of Frobenius Theorem.

Theorem 10.11 *Let M be a smooth manifold of dimension n. For every r-dimensional smooth, involutive distribution, D, on M, the family of all maximal, connected, integral manifolds of D forms a foliation of M.*

Proof. The proof of Theorem 10.11 can be found in Lee [98] (Theorem 19.21). $\square$

Chapter 11

Connections and Curvature in Vector Bundles

11.1 Connections and Connection Forms in Vector Bundles and Riemannian Manifolds

Given a manifold, M, in general, for any two points, $p, q \in M$, there is no "natural" isomorphism between the tangent spaces T_pM and T_qM. More generally, given any vector bundle, $\xi = (E, \pi, B, V)$, for any two points, $p, q \in B$, there is no "natural" isomorphism between the fibres, $E_p = \pi^{-1}(p)$ and $E_q = \pi^{-1}(q)$. Given a curve, $c\colon [0,1] \to M$, on M (resp. a curve, $c\colon [0,1] \to E$, on B), as $c(t)$ moves on M (resp. on B), how does the tangent space, $T_{c(t)}M$ (resp. the fibre $E_{c(t)} = \pi^{-1}(c(t))$) change as $c(t)$ moves?

If $M = \mathbb{R}^n$, then the spaces $T_{c(t)}\mathbb{R}^n$ are canonically isomorphic to $\mathbb{R}^n$ and any vector, $v \in T_{c(0)}\mathbb{R}^n \cong \mathbb{R}^n$, is simply moved along c by *parallel transport*, that it, at $c(t)$, the tangent vector, v, also belongs to $T_{c(t)}\mathbb{R}^n$. However, if M is curved, for example, a sphere, then it is not obvious how to "parallel transport" a tangent vector at $c(0)$ along a curve c. A way to achieve this is to define the notion of *parallel vector field* along a curve and this, in turn, can be defined in terms of the notion of *covariant derivative* of a vector field (or covariant derivative of a section, in the case of vector bundles).

Assume for simplicity that M is a surface in $\mathbb{R}^3$. Given any two vector fields, X and Y defined on some open subset, $U \subseteq \mathbb{R}^3$, for every $p \in U$, the *directional derivative, $D_XY(p)$, of Y with respect to X* is defined by

$$D_XY(p) = \lim_{t\to 0} \frac{Y(p + tX(p)) - Y(p)}{t}.$$

If $f\colon U \to \mathbb{R}$ is a differentiable function on U, for every $p \in U$, the *directional derivative, $X[f](p)$ (or $X(f)(p)$), of f with respect to X* is defined by

$$X[f](p) = \lim_{t\to 0} \frac{f(p + tX(p)) - f(p)}{t}.$$

We know that $X[f](p) = df_p(X(p))$.

It is easily shown that $D_X Y(p)$ is $\mathbb{R}$-bilinear in X and Y, is $C^\infty(U)$-linear in X and satisfies the Leibnitz derivation rule with respect to Y, that is:

Proposition 11.1 *The directional derivative of vector fields satisfies the following properties:*

$$\begin{aligned} D_{X_1+X_2}Y(p) &= D_{X_1}Y(p) + D_{X_2}Y(p) \\ D_{fX}Y(p) &= fD_XY(p) \\ D_X(Y_1+Y_2)(p) &= D_XY_1(p) + D_XY_2(p) \\ D_X(fY)(p) &= X[f](p)Y(p) + f(p)D_XY(p), \end{aligned}$$

for all $X, X_1, X_2, Y, Y_1, Y_2 \in \mathfrak{X}(U)$ *and all* $f \in C^\infty(U)$.

Now, if $p \in U$ where $U \subseteq M$ is an open subset of M, for any vector field, Y, defined on U ($Y(p) \in T_pM$, for all $p \in U$), for every $X \in T_pM$, the directional derivative, $D_XY(p)$, makes sense and it has an orthogonal decomposition,

$$D_XY(p) = \nabla_X Y(p) + (D_n)_X Y(p),$$

where its *horizontal (or tangential) component* is $\nabla_X Y(p) \in T_pM$ and its normal component is $(D_n)_X Y(p)$. The component, $\nabla_X Y(p)$, is the *covariant derivative* of Y with respect to $X \in T_pM$ and it allows us to define the covariant derivative of a vector field, $Y \in \mathfrak{X}(U)$, with respect to a vector field, $X \in \mathfrak{X}(M)$, on M. We easily check that $\nabla_X Y$ satisfies the four equations of Proposition 11.1.

In particular, Y, may be a vector field associated with a curve, $c\colon [0,1] \to M$. A *vector field along a curve, c,* is a vector field, Y, such that $Y(c(t)) \in T_{c(t)}M$, for all $t \in [0,1]$. We also write $Y(t)$ for $Y(c(t))$. Then, we say that Y *is parallel along* c iff $\nabla_{\partial/\partial t}Y = 0$ along c.

The notion of *parallel transport* on a surface can be defined using parallel vector fields along curves. Let p, q be any two points on the surface M and assume there is a curve, $c\colon [0,1] \to M$, joining $p = c(0)$ to $q = c(1)$. Then, using the uniqueness and existence theorem for ordinary differential equations, it can be shown that for any initial tangent vector, $Y_0 \in T_pM$, there is a unique parallel vector field, Y, along c, with $Y(0) = Y_0$. If we set $Y_1 = Y(1)$, we obtain a linear map, $Y_0 \mapsto Y_1$, from T_pM to T_qM which is also an isometry.

As a summary, given a surface, M, if we can define a notion of covariant derivative, $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, satisfying the properties of Proposition 11.1, then we can define the notion of parallel vector field along a curve and the notion of parallel transport, which yields a natural way of relating two tangent spaces, T_pM and T_qM, using curves joining p and q. This can be generalized to manifolds and even to vector bundles using the notion of connection. We will see that the notion of connection induces the notion of curvature.

Moreover, if M has a Riemannian metric, we will see that this metric induces a unique connection with two extra properties (the *Levi-Civita* connection).

Given a manifold, M, as $\mathfrak{X}(M) = \Gamma(M, TM) = \Gamma(TM)$, the set of smooth sections of the tangent bundle, TM, it is natural that for a vector bundle, $\xi = (E, \pi, B, V)$, a connection on ξ should be some kind of bilinear map,

$$\mathfrak{X}(B) \times \Gamma(\xi) \longrightarrow \Gamma(\xi),$$

that tells us how to take the covariant derivative of sections.

Technically, it turns out that it is cleaner to define a connection on a vector bundle, ξ, as an $\mathbb{R}$-linear map,

$$\nabla\colon \Gamma(\xi) \to \mathcal{A}^1(B) \otimes_{C^\infty(B)} \Gamma(\xi), \tag{$*$}$$

that satisfies the "Leibniz rule"

$$\nabla(fs) = df \otimes s + f\nabla s,$$

with $s \in \Gamma(\xi)$ and $f \in C^\infty(B)$, where $\Gamma(\xi)$ and $\mathcal{A}^1(B)$ are treated as $C^\infty(B)$-modules. Since $\mathcal{A}^1(B) = \Gamma(B, T^*B) = \Gamma(T^*B)$ and, by Proposition 7.12,

$$\begin{aligned} \mathcal{A}^1(B) \otimes_{C^\infty(B)} \Gamma(\xi) &= \Gamma(T^*B) \otimes_{C^\infty(B)} \Gamma(\xi) \\ &\cong \Gamma(T^*B \otimes \xi) \\ &\cong \Gamma(\mathcal{H}om(TB, \xi)) \\ &\cong \mathrm{Hom}_{C^\infty(B)}(\Gamma(TB), \Gamma(\xi)) \\ &= \mathrm{Hom}_{C^\infty(B)}(\mathfrak{X}(B), \Gamma(\xi)), \end{aligned}$$

the range of ∇ can be viewed as a space of $\Gamma(\xi)$-valued differential forms on B. Milnor and Stasheff [110] (Appendix C) use the version where

$$\nabla\colon \Gamma(\xi) \to \Gamma(T^*B \otimes \xi)$$

and Madsen and Tornehave [100] (Chapter 17) use the equivalent version stated in $(*)$. A thorough presentation of connections on vector bundles and the various ways to define them can be found in Postnikov [125] which also constitutes one of the most extensive references on differential geometry. Set

$$\mathcal{A}^1(\xi) = \mathcal{A}^1(B; \xi) = \mathcal{A}^1(B) \otimes_{C^\infty(B)} \Gamma(\xi)$$

and, more generally, for any $i \geq 0$, set

$$\mathcal{A}^i(\xi) = \mathcal{A}^i(B; \xi) = \mathcal{A}^i(B) \otimes_{C^\infty(B)} \Gamma(\xi) \cong \Gamma\Big(\Big(\bigwedge^i T^*B\Big) \otimes \xi\Big).$$

Obviously, $\mathcal{A}^0(\xi) = \Gamma(\xi)$ (and recall that $\mathcal{A}^0(B) = C^\infty(B)$). The space of differential forms, $\mathcal{A}^i(B; \xi)$, with values in $\Gamma(\xi)$ is a generalization of the space, $\mathcal{A}^i(M, F)$, of differential forms with values in F encountered in Section 8.4.

If we use the isomorphism

$$\mathcal{A}^1(B) \otimes_{C^\infty(B)} \Gamma(\xi) \cong \mathrm{Hom}_{C^\infty(B)}(\mathfrak{X}(B), \Gamma(\xi)),$$

then a connection is an $\mathbb{R}$-linear map,

$$\nabla \colon \Gamma(\xi) \longrightarrow \mathrm{Hom}_{C^\infty(B)}(\mathfrak{X}(B), \Gamma(\xi)),$$

satisfying a Leibnitz-type rule or equivalently, an $\mathbb{R}$-bilinear map,

$$\nabla \colon \mathfrak{X}(B) \times \Gamma(\xi) \longrightarrow \Gamma(\xi),$$

such that, for any $X \in \mathfrak{X}(B)$ and $s \in \Gamma(\xi)$, if we write $\nabla_X s$ instead of $\nabla(X, s)$, then the following properties hold for all $f \in C^\infty(B)$:

$$\begin{aligned} \nabla_{fX} s &= f \nabla_X s \\ \nabla_X (fs) &= X[f]s + f \nabla_X s. \end{aligned}$$

This second version may be considered simpler than the first since it does not involve a tensor product. Since

$$\mathcal{A}^1(B) = \Gamma(T^*B) \cong \mathrm{Hom}_{C^\infty(B)}(\mathfrak{X}(B), C^\infty(B)) = \mathfrak{X}(B)^*,$$

using Proposition 22.36, the isomorphism

$$\alpha \colon \mathcal{A}^1(B) \otimes_{C^\infty(B)} \Gamma(\xi) \cong \mathrm{Hom}_{C^\infty(B)}(\mathfrak{X}(B), \Gamma(\xi))$$

can be described in terms of the evaluation map,

$$\mathrm{Ev}_X \colon \mathcal{A}^1(B) \otimes_{C^\infty(B)} \Gamma(\xi) \to \Gamma(\xi),$$

given by

$$\mathrm{Ev}_X(\omega \otimes s) = \omega(X)s, \qquad X \in \mathfrak{X}(B),\ \omega \in \mathcal{A}^1(B),\ s \in \Gamma(\xi).$$

Namely, for any $\theta \in \mathcal{A}^1(B) \otimes_{C^\infty(B)} \Gamma(\xi)$,

$$\alpha(\theta)(X) = \mathrm{Ev}_X(\theta).$$

In particular, the reader should check that

$$\mathrm{Ev}_X(df \otimes s) = X[f]s.$$

Then, it is easy to see that we pass from the first version of ∇, where

$$\nabla \colon \Gamma(\xi) \to \mathcal{A}^1(B) \otimes_{C^\infty(B)} \Gamma(\xi) \tag{$*$}$$

with the Leibnitz rule

$$\nabla(fs) = df \otimes s + f\nabla s,$$

to the second version of ∇, denoted ∇', where

$$\nabla' \colon \mathfrak{X}(B) \times \Gamma(\xi) \to \Gamma(\xi), \tag{$**$}$$

is $\mathbb{R}$-bilinear and where the two conditions

$$\begin{aligned} \nabla'_{fX} s &= f \nabla'_X s \\ \nabla'_X(fs) &= X[f]s + f\nabla'_X s \end{aligned}$$

hold, *via* the equation

$$\nabla'_X = \mathrm{Ev}_X \circ \nabla.$$

From now on, we will simply write $\nabla_X s$ instead of $\nabla'_X s$, unless confusion arise. As summary of the above discussion, we make the following definition:

Definition 11.1 Let $\xi = (E, \pi, B, V)$ be a smooth real vector bundle. A *connection* on ξ is an $\mathbb{R}$-linear map,

$$\nabla \colon \Gamma(\xi) \to \mathcal{A}^1(B) \otimes_{C^\infty(B)} \Gamma(\xi) \tag{$*$}$$

such that the *Leibniz rule*

$$\nabla(fs) = df \otimes s + f\nabla s$$

holds, for all $s \in \Gamma(\xi)$ and all $f \in C^\infty(B)$. For every $X \in \mathfrak{X}(B)$, we let

$$\nabla_X = \mathrm{Ev}_X \circ \nabla$$

and for every $s \in \Gamma(\xi)$, we call $\nabla_X s$ the *covariant derivative of s relative to X*. Then, the family, (∇_X), induces a $\mathbb{R}$-bilinear map also denoted ∇,

$$\nabla \colon \mathfrak{X}(B) \times \Gamma(\xi) \to \Gamma(\xi), \tag{$**$}$$

such that the following two conditions hold:

$$\begin{aligned} \nabla_{fX} s &= f \nabla_X s \\ \nabla_X(fs) &= X[f]s + f\nabla_X s, \end{aligned}$$

for all $s \in \Gamma(\xi)$, all $X \in \mathfrak{X}(B)$ and all $f \in C^\infty(B)$. We refer to $(*)$ as the *first version* of a connection and to $(**)$ as the *second version* of a connection.

Observe that in terms of the $\mathcal{A}^i(\xi)$'s, a connection is a linear map,

$$\nabla \colon \mathcal{A}^0(\xi) \to \mathcal{A}^1(\xi),$$

satisfying the Leibniz rule. When $\xi = TB$, a connection (second version) is what is known as an *affine connection* on a manifold, B.

Remark: Given two connections, ∇^1 and ∇^2, we have

$$\nabla^1(fs) - \nabla^2(fs) = df \otimes s + f\nabla^1 s - df \otimes s - f\nabla^2 s = f(\nabla^1 s - \nabla^2 s),$$

which shows that $\nabla^1 - \nabla^2$ is a $C^\infty(B)$-linear map from $\Gamma(\xi)$ to $\mathcal{A}^1(B) \otimes_{C^\infty(B)} \Gamma(\xi)$. However

$$\begin{aligned} \mathrm{Hom}_{C^\infty(B)}(\mathcal{A}^0(\xi), \mathcal{A}^i(\xi)) &= \mathrm{Hom}_{C^\infty(B)}(\Gamma(\xi), \mathcal{A}^i(B) \otimes_{C^\infty(B)} \Gamma(\xi)) \\ &\cong \Gamma(\xi)^* \otimes_{C^\infty(B)} (\mathcal{A}^i(B) \otimes_{C^\infty(B)} \Gamma(\xi)) \\ &\cong \mathcal{A}^i(B) \otimes_{C^\infty(B)} (\Gamma(\xi)^* \otimes_{C^\infty(B)} \Gamma(\xi)) \\ &\cong \mathcal{A}^i(B) \otimes_{C^\infty(B)} \mathrm{Hom}_{C^\infty(B)}(\Gamma(\xi), \Gamma(\xi)) \\ &\cong \mathcal{A}^i(B) \otimes_{C^\infty(B)} \Gamma(\mathcal{H}om(\xi, \xi)) \\ &= \mathcal{A}^i(\mathcal{H}om(\xi, \xi)). \end{aligned}$$

Therefore, $\nabla^1 - \nabla^2 \in \mathcal{A}^1(\mathcal{H}om(\xi, \xi))$, that is, it is a one-form with values in $\Gamma(\mathcal{H}om(\xi, \xi))$. But then, the vector space, $\Gamma(\mathcal{H}om(\xi, \xi))$, acts on the space of connections (by addition) and makes the space of connections into an affine space. Given any connection, ∇ and any one-form, $\omega \in \Gamma(\mathcal{H}om(\xi, \xi))$, the expression $\nabla + \omega$ is also a connection. Equivalently, any affine combination of connections is also a connection.

A basic property of ∇ is that it is a local operator.

Proposition 11.2 *Let $\xi = (E, \pi, B, V)$ be a smooth real vector bundle and let ∇ be a connection on ξ. For every open subset, $U \subseteq B$, for every section, $s \in \Gamma(\xi)$, if $s \equiv 0$ on U, then $\nabla s \equiv 0$ on U, that is, ∇ is a local operator.*

Proof. By Proposition 3.24 applied to the constant function with value 1, for every $p \in U$, there is some open subset, $V \subseteq U$, containing p and a smooth function, $f \colon B \to \mathbb{R}$, such that $\mathrm{supp}\, f \subseteq U$ and $f \equiv 1$ on V. Consequently, fs is a smooth section which is identically zero. By applying the Leibnitz rule, we get

$$0 = \nabla(fs) = df \otimes s + f\nabla s,$$

which, evaluated at p yields $(\nabla s)(p) = 0$, since $f(p) = 1$ and $df \equiv 0$ on V. $\square$

As an immediate consequence of Proposition 11.2, if s_1 and s_2 are two sections in $\Gamma(\xi)$ that agree on U, then $s_1 - s_2$ is zero on U, so $\nabla(s_1 - s_2) = \nabla s_1 - \nabla s_2$ is zero on U, that is, ∇s_1 and ∇s_2 agree on U.

Proposition 11.2 also implies that a connection, ∇, on ξ, restricts to a connection, $\nabla \upharpoonright U$ on the vector bundle, $\xi \upharpoonright U$, for every open subset, $U \subseteq B$. Indeed, let s be a section of ξ over U. Pick any $b \in U$ and define $(\nabla s)(b)$ as follows: Using Proposition 3.24, there is some open subset, $V_1 \subseteq U$, containing b and a smooth function, $f_1 \colon B \to \mathbb{R}$, such that $\mathrm{supp}\, f_1 \subseteq U$ and $f_1 \equiv 1$ on V_1 so, let $s_1 = f_1 s$, a global section of ξ. Clearly, $s_1 = s$ on V_1, and set

$$(\nabla s)(b) = (\nabla s_1)(b).$$

This definition does not depend on (V_1, f_1), because if we had used another pair, (V_2, f_2), as above, since $b \in V_1 \cap V_2$, we have

$$s_1 = f_1 s = s = f_2 s = s_2 \qquad \text{on} \quad V_1 \cap V_2$$

so, by Proposition 11.2,

$$(\nabla s_1)(b) = (\nabla s_2)(b).$$

It should also be noted that $(\nabla_X s)(b)$ only depends on $X(b)$, that is, for any two vector fields, $X, Y \in \mathfrak{X}(B)$, if $X(b) = Y(b)$ for some $b \in B$, then

$$(\nabla_X s)(b) = (\nabla_Y s)(b), \qquad \text{for every } s \in \Gamma(\xi).$$

As above, by linearity, it it enough to prove that if $X(b) = 0$, then $(\nabla_X s)(b) = 0$. To prove this, pick any local trivialization, (U, φ), with $b \in U$. Then, we can write

$$X \upharpoonright U = \sum_{i=1}^{d} X_i \frac{\partial}{\partial x_i}.$$

However, as before, we can find a pair, (V, f), with $b \in V \subseteq U$, $\operatorname{supp} f \subseteq U$ and $f = 1$ on V, so that $f \frac{\partial}{\partial x_i}$ is a smooth vector field on B and $f \frac{\partial}{\partial x_i}$ agrees with $\frac{\partial}{\partial x_i}$ on V, for $i = 1, \ldots, n$. Clearly, $f X_i \in C^\infty(B)$ and $f X_i$ agrees with X_i on V so if we write $\widetilde{X} = f^2 X$, then

$$\widetilde{X} = f^2 X = \sum_{i=1}^{d} f X_i \, f \frac{\partial}{\partial x_i}$$

and we have

$$f^2 \nabla_X s = \nabla_{\widetilde{X}} s = \sum_{i=1}^{d} f X_i \, \nabla_{f \frac{\partial}{\partial x_i}} s.$$

Since $X_i(b) = 0$ and $f(b) = 1$, we get $(\nabla_X s)(b) = 0$, as claimed.

Using the above property, for any point, $p \in B$, we can define the covariant derivative, $(\nabla_u s)(p)$, of a section, $s \in \Gamma(\xi)$, with respect to a tangent vector, $u \in T_p B$. Indeed, pick any vector field, $X \in \mathfrak{X}(B)$, such that $X(p) = u$ (such a vector field exists locally over the domain of a chart and then extend it using a bump function) and set $(\nabla_u s)(p) = (\nabla_X s)(p)$. By the above property, if $X(p) = Y(p)$, then $(\nabla_X s)(p) = (\nabla_Y s)(p)$ so $(\nabla_u s)(p)$ is well-defined. Since ∇ is a local operator, $(\nabla_u s)(p)$ is also well defined for any tangent vector, $u \in T_p B$, and any local section, $s \in \Gamma(U, \xi)$, defined in some open subset, U, with $p \in U$. From now on, we will use this property without any further justification.

Since ξ is locally trivial, it is interesting to see what $\nabla \upharpoonright U$ looks like when (U, φ) is a local trivialization of ξ.

Fix once and for all some basis, $(v_1, \ldots, v_n)$, of the typical fibre, V ($n = \dim(V)$). To every local trivialization, $\varphi\colon \pi^{-1}(U) \to U \times V$, of ξ (for some open subset, $U \subseteq B$), we associate the frame, $(s_1, \ldots, s_n)$, over U given by

$$s_i(b) = \varphi^{-1}(b, v_i), \qquad b \in U.$$

Then, every section, s, over U, can be written uniquely as $s = \sum_{i=1}^n f_i s_i$, for some functions $f_i \in C^\infty(U)$ and we have

$$\nabla s = \sum_{i=1}^n \nabla(f_i s_i) = \sum_{i=1}^n (df_i \otimes s_i + f_i \nabla s_i).$$

On the other hand, each ∇s_i can be written as

$$\nabla s_i = \sum_{j=1}^n \omega_{ij} \otimes s_j,$$

for some $n \times n$ matrix, $\omega = (\omega_{ij})$, of one-forms, $\omega_{ij} \in \mathcal{A}^1(U)$, so we get

$$\nabla s = \sum_{i=1}^n df_i \otimes s_i + \sum_{i=1}^n f_i \nabla s_i = \sum_{i=1}^n df_i \otimes s_i + \sum_{i,j=1}^n f_i \omega_{ij} \otimes s_j = \sum_{j=1}^n (df_j + \sum_{i=1}^n f_i \omega_{ij}) \otimes s_j.$$

With respect to the frame, $(s_1, \ldots, s_n)$, the connection ∇ has the matrix form

$$\nabla(f_1, \ldots, f_n) = (df_1, \ldots, df_n) + (f_1, \ldots, f_n)\omega$$

and the matrix, $\omega = (\omega_{ij})$, of one-forms, $\omega_{ij} \in \mathcal{A}^1(U)$, is called the *connection form* or *connection matrix* of ∇ with respect to $\varphi\colon \pi^{-1}(U) \to U \times V$. The above computation also shows that on U, any connection is uniquely determined by a matrix of one-forms, $\omega_{ij} \in \mathcal{A}^1(U)$. In particular, the connection on U for which

$$\nabla s_1 = 0, \ldots, \nabla s_n = 0,$$

corresponding to the zero matrix is called the *flat connection on U* (w.r.t. $(s_1, \ldots, s_n)$).

Some authors (such as Morita [114]) use a notation involving subscripts and superscripts, namely

$$\nabla s_i = \sum_{j=1}^n \omega_i^j \otimes s_j.$$

But, beware, the expression $\omega = (\omega_i^j)$ denotes the $n \times n$-matrix whose rows are indexed by j and whose columns are indexed by i! Accordingly, if $\theta = \omega\eta$, then

$$\theta_j^i = \sum_k \omega_k^i \eta_j^k.$$

The matrix, (ω_j^i) is thus *the transpose* of our matrix (ω_{ij}). This has the effects that some of the results differ either by a sign (as in $\omega \wedge \omega$) or by a permutation of matrices (as in the formula for a change of frame).

Remark: If $(\theta_1, \ldots, \theta_n)$ is the dual frame of $(s_1, \ldots, s_n)$, that is, $\theta_i \in \mathcal{A}^1(U)$, is the one-form defined so that

$$\theta_i(b)(s_j(b)) = \delta_{ij}, \qquad \text{for all} \quad b \in U,\ 1 \leq i, j \leq n,$$

then we can write $\omega_{ik} = \sum_{j=1}^{n} \Gamma^k_{ji} \theta_j$ and so,

$$\nabla s_i = \sum_{j,k=1}^{n} \Gamma^k_{ji} (\theta_j \otimes s_k),$$

where the $\Gamma^k_{ji} \in C^\infty(U)$ are the *Christoffel symbols*.

Proposition 11.3 *Every vector bundle, ξ, possesses a connection.*

Proof. Since ξ is locally trivial, we can find a locally finite open cover, $(U_\alpha)_\alpha$, of B such that $\pi^{-1}(U_\alpha)$ is trivial. If (f_α) is a partition of unity subordinate to the cover $(U_\alpha)_\alpha$ and if ∇^α is any flat connection on $\xi \restriction U_\alpha$, then it is immediately verified that

$$\nabla = \sum_\alpha f_\alpha \nabla^\alpha$$

is a connection on ξ. $\square$

If $\varphi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times V$ and $\varphi_\beta \colon \pi^{-1}(U_\beta) \to U_\beta \times V$ are two overlapping trivializations, we know that for every $b \in U_\alpha \cap U_\beta$, we have

$$\varphi_\alpha \circ \varphi_\beta^{-1}(b, u) = (b, g_{\alpha\beta}(b)u),$$

where $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(V)$ is the transition function. As

$$\varphi_\beta^{-1}(b, u) = \varphi_\alpha^{-1}(b, g_{\alpha\beta}(b)u),$$

if $(s_1, \ldots, s_n)$ is the frame over U_α associated with φ_α and $(t_1, \ldots, t_n)$ is the frame over U_β associated with φ_β, we see that

$$t_i = \sum_{j=1}^{n} g_{ij} s_j,$$

where $g_{\alpha\beta} = (g_{ij})$.

Proposition 11.4 *With the notations as above, the connection matrices, ω_α and ω_β respectively over U_α and U_β obey the tranformation rule*

$$\omega_\beta = g_{\alpha\beta} \omega_\alpha g_{\alpha\beta}^{-1} + (dg_{\alpha\beta}) g_{\alpha\beta}^{-1},$$

where $dg_{\alpha\beta} = (dg_{ij})$.

To prove the above proposition, apply ∇ to both side of the equations

$$t_i = \sum_{j=1}^{n} g_{ij} s_j$$

and use ω_α and ω_β to express ∇t_i and ∇s_j. The details are left as an exercise.

In Morita [114] (Proposition 5.22), the order of the matrices in the equation of Proposition 11.4 must be reversed.

If $\xi = TM$, the tangent bundle of some smooth manifold, M, then a connection on TM, also called a *connection on* M is a linear map,

$$\nabla\colon \mathfrak{X}(M) \longrightarrow \mathcal{A}^1(M) \otimes_{C^\infty(M)} \mathfrak{X}(M) \cong \mathrm{Hom}_{C^\infty(M)}(\mathfrak{X}(M), (\mathfrak{X}(M)),$$

since $\Gamma(TM) = \mathfrak{X}(M)$. Then, for fixed $Y \in \mathcal{X}(M)$, the map ∇Y is $C^\infty(M)$-linear, which implies that ∇Y is a $(1,1)$ tensor. In a local chart, (U, φ), we have

$$\nabla_{\frac{\partial}{\partial x_i}}\left(\frac{\partial}{\partial x_j}\right) = \sum_{k=1}^{n} \Gamma^k_{ij} \frac{\partial}{\partial x_k},$$

where the Γ^k_{ij} are Christoffel symbols.

The covariant derivative, ∇_X, given by a connection, ∇, on TM, can be extended to a covariant derivative, $\nabla^{r,s}_X$, defined on tensor fields in $\Gamma(M, T^{r,s}(M))$, for all $r, s \geq 0$, where

$$T^{r,s}(M) = T^{\otimes r}M \otimes (T^*M)^{\otimes s}.$$

We already have $\nabla^{1,0}_X = \nabla_X$ and it is natural to set $\nabla^{0,0}_X f = X[f] = df(X)$. Recall that there is an isomorphism between the set of tensor fields, $\Gamma(M, T^{r,s}(M))$, and the set of $C^\infty(M)$-multilinear maps,

$$\Phi\colon \underbrace{\mathcal{A}^1(M) \times \cdots \times \mathcal{A}^1(M)}_{r} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{s} \longrightarrow C^\infty(M),$$

where $\mathcal{A}^1(M)$ and $\mathfrak{X}(M)$ are $C^\infty(M)$-modules.

The next proposition is left as an exercise. For help, see O'Neill [119], Chapter 2, Proposition 13 and Theorem 15.

Proposition 11.5 *for every vector field, $X \in \mathfrak{X}(M)$, there is a unique family of $\mathbb{R}$-linear map, $\nabla^{r,s}\colon \Gamma(M, T^{r,s}(M)) \to \Gamma(M, T^{r,s}(M))$, with $r, s \geq 0$, such that*

(a) $\nabla^{0,0}_X f = df(X)$, *for all* $f \in C^\infty(M)$ *and* $\nabla^{1,0}_X = \nabla_X$, *for all* $X \in \mathfrak{X}(M)$.

(b) $\nabla^{r_1+r_2, s_1+s_2}_X (S \otimes T) = \nabla^{r_1,s_1}_X(S) \otimes T + S \otimes \nabla^{r_2,s_2}_X(T)$, *for all* $S \in \Gamma(M, T^{r_1,s_1}(M))$ *and all* $T \in \Gamma(M, T^{r_2,s_2}(M))$.

(c) $\nabla^{r-1,s-1}_X(c_{ij}(S)) = c_{ij}(\nabla^{r,s}_X(S))$, *for all* $S \in \Gamma(M, T^{r,s}(M))$ *and all contractions,* c_{ij}, *of* $\Gamma(M, T^{r,s}(M))$.

Furthermore,

$$(\nabla_X^{0,1}\theta)(Y) = X[\theta(Y)] - \theta(\nabla_X Y),$$

for all $X, Y \in \mathfrak{X}(M)$ and all one-forms, $\theta \in \mathcal{A}^1(M)$ and for every $S \in \Gamma(M, T^{r,s}(M))$, with $r + s \geq 2$, the covariant derivative, $\nabla_X^{r,s}(S)$, is given by

$$\begin{aligned}
(\nabla_X^{r,s} S)(\theta_1, \ldots, \theta_r, X_1, \ldots, X_s) \quad &= \quad X[S(\theta_1, \ldots, \theta_r, X_1, \ldots, X_s)] \\
&\quad - \sum_{i=1}^{r} S(\theta_1, \ldots, \nabla_X^{0,1}\theta_i, \ldots, \theta_r, X_1, \ldots, X_s) \\
&\quad - \sum_{j=1}^{s} S(\theta_1, \ldots, \ldots, \theta_r, X_1, \ldots, \nabla_X X_j, \ldots, X_s),
\end{aligned}$$

for all $X_1, \ldots, X_s \in \mathfrak{X}(M)$ and all one-forms, $\theta_1, \ldots, \theta_r \in \mathcal{A}^1(M)$.

We define the *covariant differential*, $\nabla^{r,s} S$, of a tensor, $S \in \Gamma(M, T^{r,s}(M))$, as the $(r, s+1)$-tensor given by

$$(\nabla^{r,s} S)(\theta_1, \ldots, \theta_r, X, X_1, \ldots, X_s) = (\nabla_X^{r,s} S)(\theta_1, \ldots, \theta_r, X_1, \ldots, X_s),$$

for all $X, X_j \in \mathfrak{X}(M)$ and all $\theta_i \in \mathcal{A}^1(M)$. For simplicity of notation we usually omit the superscripts r and s. In particular, for $S = g$, the Riemannian metric on M (a $(0, 2)$ tensor), we get

$$\nabla_X(g)(Y, Z) = d(g(Y, Z))(X) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z),$$

for all $X, Y, Z \in \mathfrak{X}(M)$. We will see later on that a connection on M is compatible with a metric, g, iff $\nabla_X(g) = 0$.

Everything we did in this section applies to complex vector bundles by considering complex vector spaces instead of real vector spaces, $\mathbb{C}$-linear maps instead of $\mathbb{R}$-linear map, and the space of smooth complex-valued functions, $C^\infty(B; \mathbb{C}) \cong C^\infty(B) \otimes_{\mathbb{R}} \mathbb{C}$. We also use spaces of complex-valued differentials forms,

$$\mathcal{A}^i(B; \mathbb{C}) = \mathcal{A}^i(B) \otimes_{C^\infty(B)} C^\infty(B; \mathbb{C}) \cong \Gamma\Big(\Big(\bigwedge^i T^* B\Big) \otimes \epsilon^1_{\mathbb{C}}\Big),$$

where $\epsilon^1_{\mathbb{C}}$ is the trivial complex line bundle, $B \times \mathbb{C}$, and we define $\mathcal{A}^i(\xi)$ as

$$\mathcal{A}^i(\xi) = \mathcal{A}^i(B; \mathbb{C}) \otimes_{C^\infty(B;\mathbb{C})} \Gamma(\xi).$$

A connection is a $\mathbb{C}$-linear map, $\nabla \colon \Gamma(\xi) \to \mathcal{A}^1(\xi)$, that satisfies the same Leibnitz-type rule as before. Obviously, every differential form in $\mathcal{A}^i(B; \mathbb{C})$ can be written uniquely as $\omega + i\eta$, with $\omega, \eta \in \mathcal{A}^i(B)$. The exterior differential,

$$d \colon \mathcal{A}^i(B; \mathbb{C}) \to \mathcal{A}^{i+1}(B; \mathbb{C})$$

is defined by $d(\omega + i\eta) = d\omega + i d\eta$. We obtain complex-valued de Rham cohomology groups,

$$H^i_{\mathrm{DR}}(M; \mathbb{C}) = H^i_{\mathrm{DR}}(M) \otimes_{\mathbb{R}} \mathbb{C}.$$

11.2 Curvature, Curvature Form and Curvature Matrix

If $\xi = B \times V$ is the trivial bundle and ∇ is a flat connection on ξ, we obviously have

$$\nabla_X \nabla_Y - \nabla_Y \nabla_X = \nabla_{[X,Y]},$$

where $[X, Y]$ is the Lie bracket of the vector fields X and Y. However, for general bundles and arbitrary connections, the above fails. The error term,

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

measures what's called the *curvature* of the connection. The curvature of a connection also turns up as the failure of a certain sequence involving the spaces $\mathcal{A}^i(\xi)$ to be a cochain complex. Recall that a connection on ξ is a linear map

$$\nabla \colon \mathcal{A}^0(\xi) \to \mathcal{A}^1(\xi)$$

satisfying a Leibnitz-type rule. It is natural to ask whether ∇ can be extended to a family of operators, $d^\nabla \colon \mathcal{A}^i(\xi) \to \mathcal{A}^{i+1}(\xi)$, with properties analogous to d on $\mathcal{A}^*(B)$.

This is indeed the case and we get a sequence of map,

$$0 \longrightarrow \mathcal{A}^0(\xi) \xrightarrow{\nabla} \mathcal{A}^1(\xi) \xrightarrow{d^\nabla} \mathcal{A}^2(\xi) \longrightarrow \cdots \longrightarrow \mathcal{A}^i(\xi) \xrightarrow{d^\nabla} \mathcal{A}^{i+1}(\xi) \longrightarrow \cdots,$$

but in general, $d^\nabla \circ d^\nabla = 0$ fails. In particular, $d^\nabla \circ \nabla = 0$ generally fails. The term $K^\nabla = d^\nabla \circ \nabla$ is the *curvature form (or tensor)* of the connection ∇. As we will see it yields our previous curvature, R, back.

Our next goal is to define d^∇. For this, we first define an $C^\infty(B)$-bilinear map

$$\wedge \colon \mathcal{A}^i(\xi) \times \mathcal{A}^j(\eta) \longrightarrow \mathcal{A}^{i+j}(\xi \otimes \eta)$$

as follows:

$$(\omega \otimes s) \wedge (\tau \otimes t) = (\omega \wedge \tau) \otimes (s \otimes t),$$

where $\omega \in \mathcal{A}^i(B)$, $\tau \in \mathcal{A}^j(B)$, $s \in \Gamma(\xi)$, and $t \in \Gamma(\eta)$, where we used the fact that

$$\Gamma(\xi \otimes \eta) = \Gamma(\xi) \otimes_{C^\infty(B)} \Gamma(\eta).$$

First, consider the case where $\xi = \epsilon^1 = B \times \mathbb{R}$, the trivial line bundle over B. In this case, $\mathcal{A}^i(\xi) = \mathcal{A}^i(B)$ and we have a bilinear map

$$\wedge \colon \mathcal{A}^i(B) \times \mathcal{A}^j(\eta) \longrightarrow \mathcal{A}^{i+j}(\eta)$$

given by

$$\omega \wedge (\tau \otimes t) = (\omega \wedge \tau) \otimes t.$$

For $j = 0$, we have the bilinear map

$$\wedge \colon \mathcal{A}^i(B) \times \Gamma(\eta) \longrightarrow \mathcal{A}^i(\eta)$$

given by

$$\omega \wedge t = \omega \otimes t.$$

It is clear that the bilinear map

$$\wedge \colon \mathcal{A}^r(B) \times \mathcal{A}^s(\eta) \longrightarrow \mathcal{A}^{r+s}(\eta)$$

has the following properties:

$$\begin{aligned} (\omega \wedge \tau) \wedge \theta &= \omega \wedge (\tau \wedge \theta) \\ 1 \wedge \theta &= \theta, \end{aligned}$$

for all $\omega \in \mathcal{A}^i(B), \tau \in \mathcal{A}^j(B)$, $\theta \in \mathcal{A}^k(\xi)$ and where 1 denotes the constant function in $C^\infty(B)$ with value 1.

Proposition 11.6 *For every vector bundle, ξ, for all $j \geq 0$, there is a unique $\mathbb{R}$-linear map (resp. $\mathbb{C}$-linear if ξ is a complex VB), $d^\nabla \colon \mathcal{A}^j(\xi) \to \mathcal{A}^{j+1}(\xi)$, such that*

(i) $d^\nabla = \nabla$ *for* $j = 0$.

(ii) $d^\nabla(\omega \wedge t) = d\omega \wedge t + (-1)^i \omega \wedge d^\nabla t$, *for all* $\omega \in \mathcal{A}^i(B)$ *and all* $t \in \mathcal{A}^j(\xi)$.

Proof. Recall that $\mathcal{A}^j(\xi) = \mathcal{A}^j(B) \otimes_{C^\infty(B)} \Gamma(\xi)$ and define $d^\nabla \colon \mathcal{A}^j(B) \times \Gamma(\xi) \to \mathcal{A}^{j+1}(\xi)$ by

$$d^\nabla(\omega, s) = d\omega \otimes s + (-1)^j \omega \wedge \nabla s,$$

for all $\omega \in \mathcal{A}^j(B)$ and all $s \in \Gamma(\xi)$. We claim that d^∇ induces an $\mathbb{R}$-linear map on $\mathcal{A}^j(\xi)$ but there is a complication as d^∇ is not $C^\infty(B)$-bilinear. The way around this problem is to use Proposition 22.37. For this, we need to check that d^∇ satisfies the condition of Proposition 22.37, where the right action of $C^\infty(B)$ on $\mathcal{A}^j(B)$ is equal to the left action, namely wedging:

$$f \wedge \omega = \omega \wedge f \qquad f \in C^\infty(B) = \mathcal{A}^0(B),\ \omega \in \mathcal{A}^j(B).$$

As $\wedge$ is $C^\infty(B)$-bilinear and $\tau \otimes s = \tau \wedge s$ for all $\tau \in \mathcal{A}^i(B)$ and all $s \in \Gamma(\xi)$, we have

$$\begin{aligned} d^\nabla(\omega f, s) &= d(\omega f) \otimes s + (-1)^j (\omega f) \wedge \nabla s \\ &= d(\omega f) \wedge s + (-1)^j f\omega \wedge \nabla s \\ &= ((d\omega) f + (-1)^j \omega \wedge df) \wedge s + (-1)^j f\omega \wedge \nabla s \\ &= f d\omega \wedge s + (-1)^j \omega \wedge df \wedge s + (-1)^j f\omega \wedge \nabla s \end{aligned}$$

and

$$\begin{aligned}
d^{\nabla}(\omega, fs) &= d\omega \otimes (fs) + (-1)^j \omega \wedge \nabla(fs) \\
&= d\omega \wedge (fs) + (-1)^j \omega \wedge \nabla(fs) \\
&= f d\omega \wedge s + (-1)^j \omega \wedge (df \otimes s + f\nabla s) \\
&= f d\omega \wedge s + (-1)^j \omega \wedge (df \wedge s + f\nabla s) \\
&= f d\omega \wedge s + (-1)^j \omega \wedge df \wedge s + (-1)^j f\omega \wedge \nabla s.
\end{aligned}$$

Thus, $d^{\nabla}(\omega f, s) = d^{\nabla}(\omega, fs)$, and Proposition 22.37 shows that $d^{\nabla}\colon \mathcal{A}^j(\xi) \to \mathcal{A}^{j+1}(\xi)$ is a well-defined $\mathbb{R}$-linear map for all $j \geq 0$. Furthermore, it is clear that $d^{\nabla} = \nabla$ for $j = 0$. Now, for $\omega \in \mathcal{A}^i(B)$ and $t = \tau \otimes s \in \mathcal{A}^j(\xi)$ we have

$$\begin{aligned}
d^{\nabla}(\omega \wedge (\tau \otimes s)) &= d^{\nabla}((\omega \wedge \tau) \otimes s)) \\
&= d(\omega \wedge \tau) \otimes s + (-1)^{i+j}(\omega \wedge \tau) \wedge \nabla s \\
&= (d\omega \wedge \tau) \otimes s + (-1)^i(\omega \wedge d\tau) \otimes s + (-1)^{i+j}(\omega \wedge \tau) \wedge \nabla s \\
&= d\omega \wedge (\tau \otimes s + (-1)^i \omega \wedge (d\tau \otimes s) + (-1)^{i+j}\omega \wedge (\tau \wedge \nabla s) \\
&= d\omega \wedge (\tau \otimes s) + (-1)^i \omega \wedge d^{\nabla}(\tau \wedge s), \\
&= d\omega \wedge (\tau \otimes s) + (-1)^i \omega \wedge d^{\nabla}(\tau \otimes s),
\end{aligned}$$

which proves (ii). $\square$

As a consequence, we have the following sequence of linear maps:

$$0 \longrightarrow \mathcal{A}^0(\xi) \xrightarrow{\nabla} \mathcal{A}^1(\xi) \xrightarrow{d^{\nabla}} \mathcal{A}^2(\xi) \longrightarrow \cdots \longrightarrow \mathcal{A}^i(\xi) \xrightarrow{d^{\nabla}} \mathcal{A}^{i+1}(\xi) \longrightarrow \cdots .$$

but in general, $d^{\nabla} \circ d^{\nabla} = 0$ fails. Although generally $d^{\nabla} \circ \nabla = 0$ fails, the map $d^{\nabla} \circ \nabla$ is $C^{\infty}(B)$-linear. Indeed,

$$\begin{aligned}
(d^{\nabla} \circ \nabla)(fs) &= d^{\nabla}(df \otimes s + f\nabla s) \\
&= d^{\nabla}(df \wedge s + f \wedge \nabla s) \\
&= ddf \wedge s - df \wedge \nabla s + df \wedge \nabla s + f \wedge d^{\nabla}(\nabla s) \\
&= f(d^{\nabla} \circ \nabla)(s)).
\end{aligned}$$

Therefore, $d^{\nabla} \circ \nabla\colon \mathcal{A}^0(\xi) \to \mathcal{A}^2(\xi)$ is a $C^{\infty}(B)$-linear map. However, recall that just before Proposition 11.2 we showed that

$$\mathrm{Hom}_{C^{\infty}(B)}(\mathcal{A}^0(\xi), \mathcal{A}^i(\xi)) \cong \mathcal{A}^i(\mathcal{H}om(\xi, \xi)),$$

therefore, $d^{\nabla} \circ \nabla \in \mathcal{A}^2(\mathcal{H}om(\xi, \xi))$, that is, $d^{\nabla} \circ \nabla$ is a two-form with values in $\Gamma(\mathcal{H}om(\xi, \xi))$.

Definition 11.2 For any vector bundle, ξ, and any connection, ∇, on ξ, the vector-valued two-form, $R^{\nabla} = d^{\nabla} \circ \nabla \in \mathcal{A}^2(\mathcal{H}om(\xi, \xi))$ is the *curvature form* (or *curvature tensor*) of the connection ∇. We say that ∇ is a *flat connection* iff $R^{\nabla} = 0$.

For simplicity of notation, we also write R for R^∇. The expression R^∇ is also denoted F^∇ or K^∇. As in the case of a connection, we can express R^∇ locally in any local trivialization, $\varphi\colon \pi^{-1}(U) \to U \times V$, of ξ. Since $R^\nabla = d^\nabla \circ \nabla \in \mathcal{A}^2(\xi) = \mathcal{A}^j(B) \otimes_{C^\infty(B)} \Gamma(\xi)$, if $(s_1, \ldots, s_n)$ is the frame associated with (φ, U), then

$$R^\nabla(s_i) = \sum_{j=1}^n \Omega_{ij} \otimes s_j,$$

for some matrix, $\Omega = (\Omega_{ij})$, of two forms, $\Omega_{ij} \in \mathcal{A}^2(B)$. We call $\Omega = (\Omega_{ij})$ the *curvature matrix* (or *curvature form*) associated with the local trivialization. The relationship between the connection form, ω, and the curvature form, Ω, is simple:

Proposition 11.7 *(Structure Equations) Let ξ be any vector bundle and let ∇ be any connection on ξ. For every local trivialization, $\varphi\colon \pi^{-1}(U) \to U \times V$, the connection matrix, $\omega = (\omega_{ij})$, and the curvature matrix, $\Omega = (\Omega_{ij})$, associated with the local trivialization, (φ, U), are related by the* **structure equation***:*

$$\Omega = d\omega - \omega \wedge \omega.$$

Proof. By definition,

$$\nabla(s_i) = \sum_{j=1}^n \omega_{ij} \otimes s_j,$$

so if we apply d^∇ and use property (ii) of Proposition 11.6 we get

$$\begin{aligned}
d^\nabla(\nabla(s_i)) &= \sum_{k=1}^n \Omega_{ik} \otimes s_k \\
&= \sum_{j=1}^n d^\nabla(\omega_{ij} \otimes s_j) \\
&= \sum_{j=1}^n d\omega_{ij} \otimes s_j - \sum_{j=1}^n \omega_{ij} \wedge \nabla s_j \\
&= \sum_{j=1}^n d\omega_{ij} \otimes s_j - \sum_{j=1}^n \omega_{ij} \wedge \sum_{k=1}^n \omega_{jk} \otimes s_k \\
&= \sum_{k=1}^n d\omega_{ik} \otimes s_k - \sum_{k=1}^n \Big(\sum_{j=1}^n \omega_{ij} \wedge \omega_{jk}\Big) \otimes s_k,
\end{aligned}$$

and so,

$$\Omega_{ik} = d\omega_{ik} - \sum_{j=1}^n \omega_{ij} \wedge \omega_{jk},$$

which, means that

$$\Omega = d\omega - \omega \wedge \omega,$$

as claimed. $\square$

Some other texts, including Morita [114] (Theorem 5.21) state the structure equations as
$$\Omega = d\omega + \omega \wedge \omega.$$

Although this is far from obvious from Definition 11.2, the curvature form, R^{∇}, is related to the curvature, $R(X, Y)$, defined at the beginning of Section 11.2. For this, we define the evaluation map

$$\mathrm{Ev}_{X,Y} \colon \mathcal{A}^2(\mathcal{H}om(\xi, \xi)) \to \mathcal{A}^0(\mathcal{H}om(\xi, \xi)) = \Gamma(\mathcal{H}om(\xi, \xi)),$$

as follows: For all $X, Y \in \mathfrak{X}(B)$, all $\omega \otimes h \in \mathcal{A}^2(\mathcal{H}om(\xi, \xi)) = \mathcal{A}^2(B) \otimes_{C^\infty(B)} \Gamma(\mathcal{H}om(\xi, \xi))$, set
$$\mathrm{Ev}_{X,Y}(\omega \otimes h) = \omega(X, Y)h.$$
It is clear that this map is $C^\infty(B)$-linear and thus well-defined on $\mathcal{A}^2(\mathcal{H}om(\xi, \xi))$. (Recall that $\mathcal{A}^0(\mathcal{H}om(\xi, \xi)) = \Gamma(\mathcal{H}om(\xi, \xi)) = \mathrm{Hom}_{C^\infty(B)}(\Gamma(\xi), \Gamma(\xi))$.) We write

$$R^{\nabla}_{X,Y} = \mathrm{Ev}_{X,Y}(R^{\nabla}) \in \mathrm{Hom}_{C^\infty(B)}(\Gamma(\xi), \Gamma(\xi)).$$

Proposition 11.8 *For any vector bundle, ξ, and any connection, ∇, on ξ, for all $X, Y \in \mathfrak{X}(B)$, if we let*
$$R(X, Y) = \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]},$$
then
$$R(X, Y) = R^{\nabla}_{X,Y}.$$

Sketch of Proof. First, check that $R(X, Y)$ is $C^\infty(B)$-linear and then work locally using the frame associated with a local trivialization using Proposition 11.7. □

Remark: Proposition 11.8 implies that $R(Y, X) = -R(X, Y)$ and that $R(X, Y)(s)$ is $C^\infty(B)$-linear in X, Y and s.

If $\varphi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times V$ and $\varphi_\beta \colon \pi^{-1}(U_\beta) \to U_\beta \times V$ are two overlapping trivializations, the relationship between the curvature matrices Ω_α and Ω_β, is given by the following proposition which is the counterpart of Proposition 11.4 for the curvature matrix:

Proposition 11.9 *If $\varphi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times V$ and $\varphi_\beta \colon \pi^{-1}(U_\beta) \to U_\beta \times V$ are two overlapping trivializations of a vector bundle, ξ, then we have the following transformation rule for the curvature matrices Ω_α and Ω_β:*
$$\Omega_\beta = g_{\alpha\beta} \Omega_\alpha g_{\alpha\beta}^{-1},$$
where $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(V)$ is the transition function.

Proof Sketch. Use the structure equations (Proposition 11.7) and apply d to the equations of Proposition 11.4. □

Proposition 11.7 also yields a formula for $d\Omega$, know as *Bianchi's identity* (in local form).

Proposition 11.10 *(Bianchi's Identity) For any vector bundle, ξ, any connection, ∇, on ξ, if ω and Ω are respectively the connection matrix and the curvature matrix, in some local trivialization, then*

$$d\Omega = \omega \wedge \Omega - \Omega \wedge \omega.$$

Proof. If we apply d to the structure equation, $\Omega = d\omega - \omega \wedge \omega$, we get

$$\begin{aligned} d\Omega &= dd\omega - d\omega \wedge \omega + \omega \wedge d\omega \\ &= -(\Omega + \omega \wedge \omega) \wedge \omega + \omega \wedge (\Omega + \omega \wedge \omega) \\ &= -\Omega \wedge \omega - \omega \wedge \omega \wedge \omega + \omega \wedge \Omega + \omega \wedge \omega \wedge \omega \\ &= \omega \wedge \Omega - \Omega \wedge \omega, \end{aligned}$$

as claimed. $\square$

We conclude this section by giving a formula for $d^{\nabla} \circ d^{\nabla}(t)$, for any $t \in \mathcal{A}^i(\xi)$. Consider the special case of the bilinear map

$$\wedge \colon \mathcal{A}^i(\xi) \times \mathcal{A}^j(\eta) \longrightarrow \mathcal{A}^{i+j}(\xi \otimes \eta)$$

defined just before Proposition 11.6 with $j = 2$ and $\eta = \mathcal{H}om(\xi, \xi)$. This is the C^{∞}-bilinear map

$$\wedge \colon \mathcal{A}^i(\xi) \times \mathcal{A}^2(\mathcal{H}om(\xi, \xi)) \longrightarrow \mathcal{A}^{i+2}(\xi \otimes \mathcal{H}om(\xi, \xi)).$$

We also have the evaluation map,

$$\begin{aligned} \mathrm{ev} \colon \mathcal{A}^j(\xi \otimes \mathcal{H}om(\xi, \xi)) \cong \mathcal{A}^j(B) \otimes_{C^{\infty}(B)} \Gamma(\xi) \otimes_{C^{\infty}(B)} \mathrm{Hom}_{C^{\infty}(B)}(\Gamma(\xi), \Gamma(\xi)) \\ \longrightarrow \mathcal{A}^j(B) \otimes_{C^{\infty}(B)} \Gamma(\xi) = \mathcal{A}^j(\xi), \end{aligned}$$

given by

$$\mathrm{ev}(\omega \otimes s \otimes h) = \omega \otimes h(s),$$

with $\omega \in \mathcal{A}^j(B)$, $s \in \Gamma(\xi)$ and $h \in \mathrm{Hom}_{C^{\infty}(B)}(\Gamma(\xi), \Gamma(\xi))$. Let

$$\wedge \colon \mathcal{A}^i(\xi) \times \mathcal{A}^2(\mathcal{H}om(\xi, \xi)) \longrightarrow \mathcal{A}^{i+2}(\xi)$$

be the composition

$$\mathcal{A}^i(\xi) \times \mathcal{A}^2(\mathcal{H}om(\xi, \xi)) \xrightarrow{\wedge} \mathcal{A}^{i+2}(\xi \otimes \mathcal{H}om(\xi, \xi)) \xrightarrow{\mathrm{ev}} \mathcal{A}^{i+2}(\xi).$$

More explicitly, the above map is given (on generators) by

$$(\omega \otimes s) \wedge H = \omega \wedge H(s),$$

where $\omega \in \mathcal{A}^i(B)$, $s \in \Gamma(\xi)$ and $H \in \mathrm{Hom}_{C^{\infty}(B)}(\Gamma(\xi), \mathcal{A}^2(\xi)) \cong \mathcal{A}^2(\mathcal{H}om(\xi, \xi))$.

Proposition 11.11 *For any vector bundle, ξ, and any connection, ∇, on ξ the composition $d^{\nabla} \circ d^{\nabla} \colon \mathcal{A}^i(\xi) \to \mathcal{A}^{i+2}(\xi)$ maps t to $t \wedge R^{\nabla}$, for any $t \in \mathcal{A}^i(\xi)$.*

Proof. Any $t \in \mathcal{A}^i(\xi)$ is some linear combination of elements $\omega \otimes s \in \mathcal{A}^i(B) \otimes_{C^\infty(B)} \Gamma(\xi)$ and by Proposition 11.6, we have

$$\begin{aligned} d^\nabla \circ d^\nabla(\omega \otimes s) &= d^\nabla(d\omega \otimes s + (-1)^i \omega \wedge \nabla s) \\ &= dd\omega \otimes s + (-1)^{i+1} d\omega \wedge \nabla s + (-1)^i d\omega \wedge \nabla s + (-1)^i(-1)^i \omega \wedge d^\nabla \circ \nabla s \\ &= \omega \wedge d^\nabla \circ \nabla s \\ &= (\omega \otimes s) \wedge R^\nabla, \end{aligned}$$

as claimed. $\square$

Proposition 11.11 shows that $d^\nabla \circ d^\nabla = 0$ iff $R^\nabla = d^\nabla \circ \nabla = 0$, that is, iff the connection ∇ is flat. Thus, the sequence

$$0 \longrightarrow \mathcal{A}^0(\xi) \xrightarrow{\nabla} \mathcal{A}^1(\xi) \xrightarrow{d^\nabla} \mathcal{A}^2(\xi) \longrightarrow \cdots \longrightarrow \mathcal{A}^i(\xi) \xrightarrow{d^\nabla} \mathcal{A}^{i+1}(\xi) \longrightarrow \cdots,$$

is a cochain complex iff ∇ is flat.

Again, everything we did in this section applies to complex vector bundles.

11.3 Parallel Transport

The notion of connection yields the notion of parallel transport in a vector bundle. First, we need to define the covariant derivative of a section along a curve.

Definition 11.3 Let $\xi = (E, \pi, B, V)$ be a vector bundle and let $\gamma \colon [a, b] \to B$ be a smooth curve in B. A *smooth section along the curve* γ is a smooth map, $X \colon [a, b] \to E$, such that $\pi(X(t)) = \gamma(t)$, for all $t \in [a, b]$. When $\xi = TB$, the tangent bundle of the manifold, B, we use the terminology *smooth vector field along* γ.

Recall that the curve $\gamma \colon [a, b] \to B$ is smooth iff γ is the restriction to $[a, b]$ of a smooth curve on some open interval containing $[a, b]$.

Proposition 11.12 *Let ξ be a vector bundle, ∇ be a connection on ξ and $\gamma \colon [a, b] \to B$ be a smooth curve in B. There is a $\mathbb{R}$-linear map, D/dt, defined on the vector space of smooth sections, X, along γ, which satisfies the following conditions:*

(1) For any smooth function, $f \colon [a, b] \to \mathbb{R}$,

$$\frac{D(fX)}{dt} = \frac{df}{dt} X + f \frac{DX}{dt}$$

(2) If X is induced by a global section, $s \in \Gamma(\xi)$, that is, if $X(t_0) = s(\gamma(t_0))$ for all $t_0 \in [a, b]$, then

$$\frac{DX}{dt}(t_0) = (\nabla_{\gamma'(t_0)}\, s)_{\gamma(t_0)}.$$

Proof. Since $\gamma([a,b])$ is compact, it can be covered by a finite number of open subsets, U_α, such that $(U_\alpha, \varphi_\alpha)$ is a local trivialization. Thus, we may assume that $\gamma\colon [a,b] \to U$ for some local trivialization, (U, φ). As $\varphi \circ \gamma\colon [a,b] \to \mathbb{R}^n$, we can write

$$\varphi \circ \gamma(t) = (u_1(t), \ldots, u_n(t)),$$

where each $u_i = pr_i \circ \varphi \circ \gamma$ is smooth. Now (see Definition 3.13), for every $g \in C^\infty(B)$, as

$$d\gamma_{t_0}\left(\frac{d}{dt}\bigg|_{t_0}\right)(g) = \frac{d}{dt}(g \circ \gamma)\bigg|_{t_0} = \frac{d}{dt}((g \circ \varphi^{-1}) \circ (\varphi \circ \gamma))\bigg|_{t_0} = \sum_{i=1}^n \frac{du_i}{dt}\left(\frac{\partial}{\partial x_i}\right)_{\gamma(t_0)} g,$$

since by definition of $\gamma'(t_0)$,

$$\gamma'(t_0) = d\gamma_{t_0}\left(\frac{d}{dt}\bigg|_{t_0}\right)$$

(see the end of Section 3.2), we have

$$\gamma'(t_0) = \sum_{i=1}^n \frac{du_i}{dt}\left(\frac{\partial}{\partial x_i}\right)_{\gamma(t_0)}.$$

If $(s_1, \ldots, s_n)$ is a frame over U, we can write

$$X(t) = \sum_{i=1}^n X_i(t) s_i(\gamma(t)),$$

for some smooth functions, X_i. Then, conditions (1) and (2) imply that

$$\frac{DX}{dt} = \sum_{j=1}^n \left(\frac{dX_j}{dt}\, s_j(\gamma(t)) + X_j(t)\nabla_{\gamma'(t)}(s_j(\gamma(t)))\right)$$

and since

$$\gamma'(t) = \sum_{i=1}^n \frac{du_i}{dt}\left(\frac{\partial}{\partial x_i}\right)_{\gamma(t)},$$

there exist some smooth functions, Γ_{ij}^k, so that

$$\nabla_{\gamma'(t)}(s_j(\gamma(t))) = \sum_{i=1}^n \frac{du_i}{dt}\,\nabla_{\frac{\partial}{\partial x_i}}(s_j(\gamma(t))) = \sum_{i,k} \frac{du_i}{dt}\,\Gamma_{ij}^k s_k(\gamma(t)).$$

It follows that

$$\frac{DX}{dt} = \sum_{k=1}^n \left(\frac{dX_k}{dt} + \sum_{ij} \Gamma_{ij}^k \frac{du_i}{dt}\, X_j\right) s_k(\gamma(t)).$$

Conversely, the above expression defines a linear operator, D/dt, and it is easy to check that it satisfies (1) and (2). $\square$

The operator, D/dt is often called *covariant derivative along* γ and it is also denoted by $\nabla_{\gamma'(t)}$ or simply $\nabla_{\gamma'}$.

Definition 11.4 Let ξ be a vector bundle and let ∇ be a connection on ξ. For every curve, $\gamma\colon [a,b] \to B$, in B, a section, X, along γ is *parallel (along γ)* iff

$$\frac{DX}{dt} = 0.$$

If M was embedded in $\mathbb{R}^d$ (for some d), then to say that X is parallel along γ would mean that the directional derivative, $(D_{\gamma'}X)(\gamma(t))$, is normal to $T_{\gamma(t)}M$.

The following proposition can be shown using the existence and uniqueness of solutions of ODE's (in our case, linear ODE's) and its proof is omitted:

Proposition 11.13 *Let ξ be a vector bundle and let ∇ be a connection on ξ. For every C^1 curve, $\gamma\colon [a,b] \to B$, in B, for every $t \in [a,b]$ and every $v \in \pi^{-1}(\gamma(t))$, there is a unique parallel section, X, along γ such that $X(t) = v$.*

For the proof of Proposition 11.13 it is sufficient to consider the portions of the curve γ contained in some local trivialization. In such a trivialization, (U, φ), as in the proof of Proposition 11.12, using a local frame, $(s_1, \ldots, s_n)$, over U, we have

$$\frac{DX}{dt} = \sum_{k=1}^{n} \left(\frac{dX_k}{dt} + \sum_{ij} \Gamma_{ij}^k \frac{du_i}{dt} X_j \right) s_k(\gamma(t)),$$

with $u_i = pr_i \circ \varphi \circ \gamma$. Consequently, X is parallel along our portion of γ iff the system of linear ODE's in the unknowns, X_k,

$$\frac{dX_k}{dt} + \sum_{ij} \Gamma_{ij}^k \frac{du_i}{dt} X_j = 0, \qquad k = 1, \ldots, n,$$

is satisfied.

Remark: Proposition 11.13 can be extended to piecewise C^1 curves.

Definition 11.5 Let ξ be a vector bundle and let ∇ be a connection on ξ. For every curve, $\gamma\colon [a,b] \to B$, in B, for every $t \in [a,b]$, the *parallel transport from $\gamma(a)$ to $\gamma(t)$ along γ* is the linear map from the fibre, $\pi^{-1}(\gamma(a))$, to the fibre, $\pi^{-1}(\gamma(t))$, which associates to any $v \in \pi^{-1}(\gamma(a))$ the vector $X_v(t) \in \pi^{-1}(\gamma(t))$, where X_v is the unique parallel section along γ with $X_v(a) = v$.

The following proposition is an immediate consequence of properties of linear ODE's:

Proposition 11.14 *Let $\xi = (E, \pi, B, V)$ be a vector bundle and let ∇ be a connection on ξ. For every C^1 curve, $\gamma\colon [a,b] \to B$, in B, the parallel transport along γ defines for every $t \in [a,b]$ a linear isomorphism, $P_\gamma\colon \pi^{-1}(\gamma(a)) \to \pi^{-1}(\gamma(t))$, between the fibres $\pi^{-1}(\gamma(a))$ and $\pi^{-1}(\gamma(t))$.*

In particular, if γ is a closed curve, that is, if $\gamma(a) = \gamma(b) = p$, we obtain a linear isomorphism, P_γ, of the fibre $E_p = \pi^{-1}(p)$, called the *holonomy of* γ. The *holonomy group of* ∇ *based at* p, denoted $\mathrm{Hol}_p(\nabla)$, is the subgroup of $\mathrm{GL}(V, \mathbb{R})$ given by

$$\mathrm{Hol}_p(\nabla) = \{P_\gamma \in \mathrm{GL}(V, \mathbb{R}) \mid \gamma \text{ is a closed curve based at } p\}.$$

If B is connected, then $\mathrm{Hol}_p(\nabla)$ depends on the basepoint $p \in B$ up to conjugation and so $\mathrm{Hol}_p(\nabla)$ and $\mathrm{Hol}_q(\nabla)$ are isomorphic for all $p, q \in B$. In this case, it makes sense to talk about the *holonomy group of* ∇. If $\xi = TB$, the tangent bundle of a manifold, B, by abuse of language, we call $\mathrm{Hol}_p(\nabla)$ the *holonomy group of* B.

11.4 Connections Compatible with a Metric; Levi-Civita Connections

If a vector bundle (or a Riemannian manifold), ξ, has a metric, then it is natural to define when a connection, ∇, on ξ is compatible with the metric. So, assume the vector bundle, ξ, has a metric, $\langle -, - \rangle$. We can use this metric to define pairings

$$\mathcal{A}^1(\xi) \times \mathcal{A}^0(\xi) \longrightarrow \mathcal{A}^1(B) \quad \text{and} \quad \mathcal{A}^0(\xi) \times \mathcal{A}^1(\xi) \longrightarrow \mathcal{A}^1(B)$$

as follows: Set (on generators)

$$\langle \omega \otimes s_1, s_2 \rangle = \langle s_1, \omega \otimes s_2 \rangle = \omega \langle s_1, s_2 \rangle,$$

for all $\omega \in \mathcal{A}^1(B)$, $s_1, s_2 \in \Gamma(\xi)$ and where $\langle s_1, s_2 \rangle$ is the function in $C^\infty(B)$ given by $b \mapsto \langle s_1(b), s_2(b) \rangle$, for all $b \in B$. More generally, we define a pairing

$$\mathcal{A}^i(\xi) \times \mathcal{A}^j(\xi) \longrightarrow \mathcal{A}^{i+j}(B),$$

by

$$\langle \omega \otimes s_1, \eta \otimes s_2 \rangle = \langle s_1, s_2 \rangle \omega \wedge \eta,$$

for all $\omega \in \mathcal{A}^i(B), \eta \in \mathcal{A}^j(B)$, $s_1, s_2 \in \Gamma(\xi)$.

Definition 11.6 Given any metric, $\langle -, - \rangle$, on a vector bundle, ξ, a connection, ∇, on ξ is *compatible with the metric*, for short, a *metric connection* iff

$$d\langle s_1, s_2 \rangle = \langle \nabla s_1, s_2 \rangle + \langle s_1, \nabla s_2 \rangle,$$

for all $s_1, s_2 \in \Gamma(\xi)$.

In terms of version-two of a connection, ∇_X is a metric connection iff

$$X(\langle s_1, s_2 \rangle) = \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle,$$

for every vector field, $X \in \mathfrak{X}(B)$.

Definition 11.6 remains unchanged if ξ is a complex vector bundle. The condition of compatibility with a metric is nicely expressed in a local trivialization. Indeed, let (U, φ) be a local trivialization of the vector bundle, ξ (of rank n). Then, using the Gram-Schmidt procedure, we obtain an orthonormal frame, $(s_1, \ldots, s_n)$, over U.

Proposition 11.15 *Using the above notations, if $\omega = (\omega_{ij})$ is the connection matrix of ∇ w.r.t. $(s_1, \ldots, s_n)$, then ω is skew-symmetric.*

Proof. Since

$$\nabla e_i = \sum_{j=1}^{n} \omega_{ij} \otimes s_j$$

and since $\langle s_i, s_j \rangle = \delta_{ij}$ (as $(s_1, \ldots, s_n)$ is orthonormal), we have $d\langle s_i, s_j \rangle = 0$ on U. Consequently

$$\begin{aligned}
0 &= d\langle s_i, s_j \rangle \\
&= \langle \nabla s_i, s_j \rangle + \langle s_i, \nabla s_j \rangle \\
&= \langle \sum_{k=1}^{n} \omega_{ik} \otimes s_k, s_j \rangle + \langle s_i, \sum_{l=1}^{n} \omega_{jl} \otimes s_l \rangle \\
&= \sum_{k=1}^{n} \omega_{ik} \langle s_k, s_j \rangle + \sum_{l=1}^{n} \omega_{jl} \langle s_i, s_l \rangle \\
&= \omega_{ij} + \omega_{ji},
\end{aligned}$$

as claimed. □

In Proposition 11.15, if ξ is a complex vector bundle, then ω is skew-Hermitian. This means that

$$\overline{\omega}^{\top} = -\omega,$$

where $\overline{\omega}$ is the conjugate matrix of ω, that is, $(\overline{\omega})_{ij} = \overline{\omega_{ij}}$. It is also easy to prove that metric connections exist.

Proposition 11.16 *Let ξ be a rank n vector with a metric, $\langle -, - \rangle$. Then, ξ, possesses metric connections.*

Proof. We can pick a locally finite cover, $(U_\alpha)_\alpha$, of B such that $(U_\alpha, \varphi_\alpha)$ is a local trivialization of ξ. Then, for each $(U_\alpha, \varphi_\alpha)$, we use the Gram-Schmidt procedure to obtain an orthonormal frame, $(s_1^\alpha, \ldots, s_n^\alpha)$, over U_α and we let ∇^α be the trivial connection on $\pi^{-1}(U_\alpha)$. By construction, ∇^α is compatible with the metric. We finish the argumemt by using a partition of unity, leaving the details to the reader. □

If ξ is a complex vector bundle, then we use a Hermitian metric and we call a connection compatible with this metric a *Hermitian connection*. In any local trivialization, the connection matrix, ω, is skew-Hermitian. The existence of Hermitian connections is clear.

If ∇ is a metric connection, then the curvature matrices are also skew-symmetric.

Proposition 11.17 *Let ξ be a rank n vector bundle with a metric, $\langle -, - \rangle$. In any local trivialization of ξ, the curvature matrix, $\Omega = (\Omega_{ij})$ is skew-symmetric. If ξ is a complex vector bundle, then $\Omega = (\Omega_{ij})$ is skew-Hermitian.*

Proof. By the structure equation (Proposition 11.7),

$$\Omega = d\omega - \omega \wedge \omega,$$

that is, $\Omega_{ij} = d\omega_{ij} - \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj}$, so, using the skew symetry of ω_{ij} and wedge,

$$\begin{aligned} \Omega_{ji} &= d\omega_{ji} - \sum_{k=1}^{n} \omega_{jk} \wedge \omega_{ki} \\ &= -d\omega_{ij} - \sum_{k=1}^{n} \omega_{kj} \wedge \omega_{ik} \\ &= -d\omega_{ij} + \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} \\ &= -\Omega_{ij}, \end{aligned}$$

as claimed. $\square$

We now restrict our attention to a Riemannian manifold, that is, to the case where our bundle, ξ, is the tangent bundle, $\xi = TM$, of some Riemannian manifold, M. We know from Proposition 11.16 that metric connections on TM exist. However, there are many metric connections on TM and none of them seems more relevant than the others. If M is a Riemannian manifold, the metric, $\langle -, - \rangle$, on M is often denoted g. In this case, for every chart, (U, φ), we let $g_{ij} \in C^\infty(M)$ be the function defined by

$$g_{ij}(p) = \left\langle \left(\frac{\partial}{\partial x_i}\right)_p, \left(\frac{\partial}{\partial x_j}\right)_p \right\rangle_p.$$

(Note the unfortunate clash of notation with the transitions functions!)

The notations $g = \sum_{ij} g_{ij} dx_i \otimes dx_j$ or simply $g = \sum_{ij} g_{ij} dx_i dx_j$ are often used to denote the metric in local coordinates. We observed immediately after stating Proposition 11.5 that the covariant differential, ∇g, of the Riemannian metric, g, on M is given by

$$\nabla_X(g)(Y, Z) = d(g(Y, Z))(X) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z),$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Therefore, a connection, ∇, on a Riemannian manifold, (M, g), is compatible with the metric iff

$$\nabla g = 0.$$

It is remarkable that if we require a certain kind of symmetry on a metric connection, then it is uniquely determined. Such a connection is known as the *Levi-Civita connection*.

The Levi-Civita connection can be characterized in several equivalent ways, a rather simple way involving the notion of torsion of a connection.

Recall that one way to introduce the curvature is to view it as the "error term"

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Another natural error term is the *torsion*, $T(X,Y)$, of the connection, ∇, given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

which measures the failure of the connection to behave like the Lie bracket.

Another way to characterize the Levi-Civita connection uses the cotangent bundle, T^*M. It turns out that a connection, ∇, on a vector bundle (metric or not), ξ, naturally induces a connection, ∇^*, on the dual bundle, ξ^*. Now, if ∇ is a connection on TM, then ∇^* is is a connection on T^*M, namely, a linear map, $\nabla^* \colon \Gamma(T^*M) \to \mathcal{A}^1(M) \otimes_{C^\infty(B)} \Gamma(T^*M)$, that is

$$\nabla^* \colon \mathcal{A}^1(M) \to \mathcal{A}^1(M) \otimes_{C^\infty(B)} \mathcal{A}^1(M) \cong \Gamma(T^*M \otimes T^*M),$$

since $\Gamma(T^*M) = \mathcal{A}^1(M)$. If we compose this map with $\wedge$, we get the map

$$\mathcal{A}^1(M) \xrightarrow{\nabla^*} \mathcal{A}^1(M) \otimes_{C^\infty(B)} \mathcal{A}^1(M) \xrightarrow{\wedge} \mathcal{A}^2(M).$$

Then, miracle, a metric connection is the Levi-Civita connection iff

$$d = \wedge \circ \nabla^*,$$

where $d \colon \mathcal{A}^1(M) \to \mathcal{A}^2(M)$ is exterior differentiation. There is also a nice local expression of the above equation.

First, we consider the definition involving the torsion.

Proposition 11.18 *(Levi-Civita, Version 1) Let M be any Riemannian manifold. There is a unique, metric, torsion-free connection, ∇, on M, that is, a connection satisfying the conditions*

$$\begin{aligned} X(\langle Y, Z\rangle) &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ \nabla_X Y - \nabla_Y X &= [X,Y], \end{aligned}$$

for all vector fields, $X, Y, Z \in \mathfrak{X}(M)$. This connection is called the **Levi-Civita connection** *(or* **canonical connection***) on M. Furthermore, this connection is determined by the* **Koszul formula**

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X(\langle Y, Z\rangle) + Y(\langle X, Z\rangle) - Z(\langle X, Y\rangle) \\ &\quad - \langle Y, [X,Z]\rangle - \langle X, [Y,Z]\rangle - \langle Z, [Y,X]\rangle. \end{aligned}$$

Proof. First, we prove uniqueness. Since our metric is a non-degenerate bilinear form, it suffices to prove the Koszul formula. As our connection is compatible with the metric, we have

$$\begin{aligned} X(\langle Y, Z\rangle) &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ Y(\langle X, Z\rangle) &= \langle \nabla_Y X, Z\rangle + \langle X, \nabla_Y Z\rangle \\ -Z(\langle X, Y\rangle) &= -\langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle \end{aligned}$$

and by adding up the above equations, we get

$$\begin{aligned} X(\langle Y, Z\rangle) + Y(\langle X, Z\rangle) - Z(\langle X, Y\rangle) &= \langle Y, \nabla_X Z - \nabla_Z X\rangle \\ &\quad + \langle X, \nabla_Y Z - \nabla_Z Y\rangle \\ &\quad + \langle Z, \nabla_X Y + \nabla_Y X\rangle. \end{aligned}$$

Then, using the fact that the torsion is zero, we get

$$\begin{aligned} X(\langle Y, Z\rangle) + Y(\langle X, Z\rangle) - Z(\langle X, Y\rangle) &= \langle Y, [X, Z]\rangle + \langle X, [Y, Z]\rangle \\ &\quad + \langle Z, [Y, X]\rangle + 2\langle Z, \nabla_X Y\rangle \end{aligned}$$

which yields the Koszul formula.

Next, we prove existence. We begin by checking that the right-hand side of the Koszul formula is $C^\infty(M)$-linear in Z, for X and Y fixed. But then, the linear map $Z \mapsto \langle \nabla_X Y, Z\rangle$ induces a one-form and $\nabla_X Y$ is the vector field corresponding to it *via* the non-degenerate pairing. It remains to check that ∇ satisfies the properties of a connection, which it a bit tedious (for example, see Kuhnel [91], Chapter 5, Section D). $\square$

Remark: In a chart, (U, φ), if we set

$$\partial_k g_{ij} = \frac{\partial}{\partial x_k}(g_{ij})$$

then it can be shown that the Christoffel symbols are given by

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{n} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}),$$

where (g^{kl}) is the inverse of the matrix (g_{kl}).

Let us now consider the second approach to torsion-freeness. For this, we have to explain how a connection, ∇, on a vector bundle, $\xi = (E, \pi, B, V)$, induces a connection, ∇^*, on the dual bundle, ξ^*. First, there is an evaluation map $\Gamma(\xi \otimes \xi^*) \longrightarrow \Gamma(\epsilon^1)$ or equivalently,

$$\langle -, -\rangle \colon \Gamma(\xi) \otimes_{C^\infty(B)} \mathrm{Hom}_{C^\infty(B)}(\Gamma(\xi), C^\infty(B)) \longrightarrow C^\infty(B),$$

given by
$$\langle s_1, s_2^* \rangle = s_2^*(s_1), \qquad s_1 \in \Gamma(\xi),\ s_2^* \in \mathrm{Hom}_{C^\infty(B)}(\Gamma(\xi), C^\infty(B))$$
and thus a map
$$\mathcal{A}^k(\xi \otimes \xi^*) = \mathcal{A}^k(B) \otimes_{C^\infty(B)} \Gamma(\xi \otimes \xi^*) \xrightarrow{\mathrm{id}\otimes\langle -,-\rangle} \mathcal{A}^k(B) \otimes_{C^\infty(B)} C^\infty(B) \cong \mathcal{A}^k(B).$$
Using this map we obtain a pairing
$$(-,-)\colon \mathcal{A}^i(\xi) \otimes \mathcal{A}^j(\xi^*) \xrightarrow{\wedge} \mathcal{A}^{i+j}(\xi \otimes \xi^*) \longrightarrow \mathcal{A}^{i+j}(B),$$
given by
$$(\omega \otimes s_1, \eta \otimes s_2^*) = (\omega \wedge \eta) \otimes \langle s_1, s_2^* \rangle,$$
where $\omega \in \mathcal{A}^i(B)$, $\eta \in \mathcal{A}^j(B)$, $s_1 \in \Gamma(\xi)$, $s_2^* \in \Gamma(\xi^*)$. It is easy to check that this pairing is non-degenerate. Then, given a connection, ∇, on a rank n vector bundle, ξ, we define ∇^* on ξ^* by
$$d\langle s_1, s_2^* \rangle = \big(\nabla(s_1), s_2^*\big) + \big(s_1, \nabla^*(s_2^*)\big),$$
where $s_1 \in \Gamma(\xi)$ and $s_2^* \in \Gamma(\xi^*)$. Because the pairing $(-,-)$ is non-degenerate, ∇^* is well-defined and it is immediately that it is a connection on ξ^*. Let us see how it is expressed locally. If (U, φ) is a local trivialization and $(s_1, \ldots, s_n)$ is the frame over U associated with (U, φ), then let $(\theta_1, \ldots, \theta_n)$ be the dual frame (called a *coframe*). We have
$$\langle s_j, \theta_i \rangle = \theta_i(s_j) = \delta_{ij}, \qquad 1 \leq i, j \leq n.$$
Recall that
$$\nabla s_j = \sum_{k=1}^n \omega_{jk} \otimes s_k$$
and write
$$\nabla^* \theta_i = \sum_{k=1}^n \omega^*_{ik} \otimes \theta_k.$$
Applying d to the equation $\langle s_j, \theta_i \rangle = \delta_{ij}$ and using the equation defining ∇^*, we get
$$\begin{aligned}
0 &= d\langle s_j, \theta_i \rangle \\
&= \big(\nabla(s_j), \theta_i\big) + \big(s_j, \nabla^*(\theta_i)\big) \\
&= \Big(\sum_{k=1}^n \omega_{jk} \otimes s_k, \theta_i\Big) + \Big(s_j, \sum_{l=1}^n \omega^*_{il} \otimes \theta_l\Big) \\
&= \sum_{k=1}^n \omega_{jk}\big(s_k, \theta_i\big) + \sum_{l=1}^n \omega^*_{il}\big(s_j, \theta_l\big) \\
&= \omega_{ji} + \omega^*_{ij}.
\end{aligned}$$

Therefore, if we write $\omega^* = (\omega^*_{ij})$, we have

$$\omega^* = -\omega^\top.$$

If ∇ is a metric connection, then ω is skew-symmetric, that is, $\omega^\top = -\omega$. In this case, $\omega^* = -\omega^\top = \omega$.

If ξ is a complex vector bundle, then there is a problem because if $(s_1, \ldots, s_n)$ is a frame over U, then the $\theta_j(b)$'s defined by

$$\langle s_i(b), \theta_j(b)\rangle = \delta_{ij}$$

are *not* linear, but instead conjugate-linear. (Recall that a linear form, θ, is *conjugate linear* (or *semi-linear*) iff $\theta(\lambda u) = \overline{\lambda}\theta(u)$, for all $\lambda \in \mathbb{C}$.) Instead of ξ^*, we need to consider the bundle $\overline{\xi}^*$, which is the bundle whose fibre over $b \in B$ consist of all conjugate-linear forms over $\pi^{-1}(b)$. In this case, the evaluation pairing, $\langle s, \theta\rangle$ is conjugate-linear in s and we find that $\omega^* = -\overline{\omega}^\top$, where ω^* is the connection matrix of $\overline{\xi}^*$ over U. If ξ is a Hermitian bundle, as ω is skew-Hermitian, we find that $\omega^* = \omega$, which makes sense since ξ and $\overline{\xi}^*$ are canonically isomorphic. However, this does not give any information on ξ^*. For this, we consider the *conjugate bundle*, $\overline{\xi}$. This is the bundle obtained from ξ by redefining the vector space structure on each fibre, $\pi^{-1}(b)$, $b \in B$, so that

$$(x + iy)v = (x - iy)v,$$

for every $v \in \pi^{-1}(b)$. If ω is the connection matrix of ξ over U, then $\overline{\omega}$ is the connection matrix of $\overline{\xi}$ over U. If ξ has a Hermitian metric, it is easy to prove that ξ^* and $\overline{\xi}$ are canonically isomorphic (see Proposition 11.32). In fact, the Hermitian product, $\langle -, -\rangle$, establishes a pairing between $\overline{\xi}$ and ξ^* and, basically as above, we can show that if $\overline{\omega}$ is the connection matrix of $\overline{\xi}$ over U, then $\omega^* = -\omega^\top$ is the the connection matrix of ξ^* over U. As ω is skew-Hermitian, $\omega^* = \overline{\omega}$.

Going back to a connection, ∇, on a manifold, M, the connection, ∇^*, is a linear map,

$$\nabla^*\colon \mathcal{A}^1(M) \longrightarrow \mathcal{A}^1(M) \otimes \mathcal{A}^1(M) \cong (\mathfrak{X}(M))^* \otimes_{C^\infty(M)} (\mathfrak{X}(M))^* \cong (\mathfrak{X}(M) \otimes_{C^\infty(M)} \mathfrak{X}(M))^*.$$

Let us figure out what $\wedge \circ \nabla^*$ is using the above interpretation. By the definition of ∇^*,

$$\nabla^*_\theta(X, Y) = X(\theta(Y)) - \theta(\nabla_X Y),$$

for every one-form, $\theta \in \mathcal{A}^1(M)$ and all vector fields, $X, Y \in \mathfrak{X}(M)$. Applying $\wedge$, we get

$$\begin{aligned}
\nabla^*_\theta(X, Y) - \nabla^*_\theta(Y, X) &= X(\theta(Y)) - \theta(\nabla_X Y) - Y(\theta(X)) + \theta(\nabla_Y X) \\
&= X(\theta(Y)) - Y(\theta(X)) - \theta(\nabla_X Y - \nabla_Y X).
\end{aligned}$$

However, recall that

$$d\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y]),$$

so we get

$$\begin{aligned}(\wedge \circ \nabla^*)(\theta)(X,Y) &= \nabla^*_\theta(X,Y) - \nabla^*_\theta(Y,X)\\ &= d\theta(X,Y) - \theta(\nabla_X Y - \nabla_Y X - [X,Y])\\ &= d\theta(X,Y) - \theta(T(X,Y)).\end{aligned}$$

It follows that for every $\theta \in \mathcal{A}^1(M)$, we have $(\wedge \circ \nabla^*)\theta = d\theta$ iff $\theta(T(X,Y)) = 0$ for all $X, Y \in \mathfrak{X}(M)$, that is iff $T(X,Y) = 0$, for all $X, Y \in \mathfrak{X}(M)$. We record this as

Proposition 11.19 *Let ξ be a manifold with connection ∇. Then, ∇ is torsion-free (i.e., $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$, for all $X, Y \in \mathfrak{X}(M)$) iff*

$$\wedge \circ \nabla^* = d,$$

where $d\colon \mathcal{A}^1(M) \to \mathcal{A}^2(M)$ is exterior differentiation.

Proposition 11.19 together with Proposition 11.18 yield a second version of the Levi-Civita Theorem:

Proposition 11.20 *(Levi-Civita, Version 2) Let M be any Riemannian manifold. There is a unique, metric connection, ∇, on M, such that*

$$\wedge \circ \nabla^* = d,$$

where $d\colon \mathcal{A}^1(M) \to \mathcal{A}^2(M)$ is exterior differentiation. This connection is equal to the Levi-Civita connection in Proposition 11.18.

Remark: If ∇ is the Levi-Civita connection of some Riemannian manifold, M, for every chart, (U, φ), we have $\omega^* = \omega$, where ω is the connection matrix of ∇ over U and ω^* is the connection matrix of the dual connection ∇^*. This implies that the Christoffel symbols of ∇ and ∇^* over U are identical. Furthermore, ∇^* is a linear map

$$\nabla^*\colon \mathcal{A}^1(M) \longrightarrow \Gamma(T^*M \otimes T^*M).$$

Thus, locally in a chart, (U, φ), if (as usual) we let $x_i = pr_i \circ \varphi$, then we can write

$$\nabla^*(dx_k) = \sum_{ij} \Gamma^k_{ij} dx_i \otimes dx_j.$$

Now, if we want $\wedge \circ \nabla^* = d$, we must have $\wedge \nabla^*(dx_k) = ddx_k = 0$, that is

$$\Gamma^k_{ij} = \Gamma^k_{ji},$$

for all i, j. Therefore, torsion-freeness can indeed be viewed as a symmetry condition on the Christoffel symbols of the connection ∇.

Our third version is a local version due to Élie Cartan. Recall that locally in a chart, (U, φ), the connection, ∇^*, is given by the matrix, ω^*, such that $\omega^* = -\omega^\top$ where ω is the connection matrix of TM over U. That is, we have

$$\nabla^* \theta_i = \sum_{j=1}^{n} -\omega_{ji} \otimes \theta_j,$$

for some one-forms, $\omega_{ij} \in \mathcal{A}^1(M)$. Then,

$$\wedge \circ \nabla^* \theta_i = -\sum_{j=1}^{n} \omega_{ji} \wedge \theta_j$$

so the requirement that $d = \wedge \circ \nabla^*$ is expressed locally by

$$d\theta_i = -\sum_{j=1}^{n} \omega_{ji} \wedge \theta_j.$$

In addition, since our connection is metric, ω is skew-symmetric and so, $\omega^* = \omega$. Then, it is not too surprising that the following proposition holds:

Proposition 11.21 *Let M be a Riemannian manifold with metric, $\langle -, - \rangle$. For every chart, (U, φ), if $(s_1, \ldots, s_n)$ is the frame over U associated with (U, φ) and $(\theta_1, \ldots, \theta_n)$ is the corresponding coframe (dual frame), then there is a unique matrix, $\omega = (\omega_{ij})$, of one-forms, $\omega_{ij} \in \mathcal{A}^1(M)$, so that the following conditions hold:*

(i) $\omega_{ji} = -\omega_{ij}$.

(ii) $d\theta_i = \sum_{j=1}^{n} \omega_{ij} \wedge \theta_j$ *or, in matrix form,* $d\theta = \omega \wedge \theta$.

Proof. There is a direct proof using a combinatorial trick, for instance, see Morita [114], Chapter 5, Proposition 5.32 or Milnor and Stasheff [110], Appendix C, Lemma 8. On the other hand, if we view $\omega = (\omega_{ij})$ as a connection matrix, then we observed that (i) asserts that the connection is metric and (ii) that it is torsion-free. We conclude by applying Proposition 11.20. □

As an example, consider an orientable (compact) surface, M, with a Riemannian metric. Pick any chart, (U, φ), and choose an orthonormal coframe of one-forms, (θ_1, θ_2), such that $\mathrm{Vol} = \theta_1 \wedge \theta_2$ on U. Then, we have

$$\begin{aligned} d\theta_1 &= a_1 \theta_1 \wedge \theta_2 \\ d\theta_2 &= a_2 \theta_1 \wedge \theta_2 \end{aligned}$$

for some functions, a_1, a_2, and we let

$$\omega_{12} = a_1\theta_1 + a_2\theta_2.$$

Clearly,

$$\begin{pmatrix} 0 & \omega_{12} \\ -\omega_{12} & 0 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 & a_1\theta_1 + a_2\theta_2 \\ -(a_1\theta_1 + a_2\theta_2) & 0 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} d\theta_1 \\ d\theta_2 \end{pmatrix}$$

which shows that

$$\omega = \omega^* = \begin{pmatrix} 0 & \omega_{12} \\ -\omega_{12} & 0 \end{pmatrix}$$

corresponds to the Levi-Civita connection on M. Let $\Omega = d\omega - \omega \wedge \omega$, we see that

$$\Omega = \begin{pmatrix} 0 & d\omega_{12} \\ -d\omega_{12} & 0 \end{pmatrix}.$$

As M is oriented and as M has a metric, the transition functions are in $\mathrm{SO}(2)$. We easily check that

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 0 & d\omega_{12} \\ -d\omega_{12} & 0 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = \begin{pmatrix} 0 & d\omega_{12} \\ -d\omega_{12} & 0 \end{pmatrix},$$

which shows that Ω is a global two-form called the *Gauss-Bonnet* 2-form of M. Then, there is a function, κ, the *Gaussian curvature of* M such that

$$d\omega_{12} = -\kappa \mathrm{Vol},$$

where Vol is the oriented volume form on M. The *Gauss-Bonnet Theorem* for orientable surfaces asserts that

$$\int_M d\omega_{12} = 2\pi\chi(M),$$

where $\chi(M)$ is the *Euler characteristic* of M.

Remark: The Levi-Civita connection induced by a Riemannian metric, g, can also be defined in terms of the Lie derivative of the metric, g. This is the approach followed in Petersen [121] (Chapter 2). If θ_X is the one-form given by

$$\theta_X = i_X g,$$

that is, $(i_X g)(Y) = g(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$ and if $L_X g$ is the Lie derivative of the symmetric $(0, 2)$ tensor, g, defined so that

$$(L_X g)(Y, Z) = X(g(Y, Z)) - g(L_X Y, Z) - g(Y, L_X Z)$$

(see Proposition 8.18), then, it is proved in Petersen [121] (Chapter 2, Theorem 1) that the Levi-Civita connection is defined implicitly by the formula

$$2g(\nabla_X Y, Z) = (L_Y g)(X, Z) + (d\theta_Y)(X, Z).$$

We conclude this section with various useful facts about torsion-free or metric connections. First, there is a nice characterization for the Levi-Civita connection induced by a Riemannian manifold over a submanifold. The proof of the next proposition is left as an exercise.

Proposition 11.22 *Let M be any Riemannian manifold and let N be any submanifold of M equipped with the induced metric. If ∇^M and ∇^N are the Levi-Civita connections on M and N, respectively, induced by the metric on M, then for any two vector fields, X and Y in $\mathfrak{X}(M)$ with $X(p), Y(p) \in T_pN$, for all $p \in N$, we have*

$$\nabla^N_X Y = (\nabla^M_X Y)^{\parallel},$$

where $(\nabla^M_X Y)^{\parallel}(p)$ is the orthogonal projection of $\nabla^M_X Y(p)$ onto T_pN, for every $p \in N$.

In particular, if γ is a curve on a surface, $M \subseteq \mathbb{R}^3$, then a vector field, $X(t)$, along γ is parallel iff $X'(t)$ is normal to the tangent plane, $T_{\gamma(t)}M$.

For any manifold, M, and any connection, ∇, on M, if ∇ is torsion-free, then the Lie derivative of any $(p, 0)$-tensor can be expressed in terms of ∇ (see Proposition 8.18).

Proposition 11.23 *For every $(0, q)$-tensor, $S \in \Gamma(M, (T^*M)^{\otimes q})$, we have*

$$(L_X S)(X_1, \ldots, X_q) = X[S(X_1, \ldots, X_q)] + \sum_{i=1}^{q} S(X_1, \ldots, \nabla_X X_i, \ldots, X_q),$$

for all $X_1, \ldots, X_q, X \in \mathfrak{X}(M)$.

Proposition 11.23 is proved in Gallot, Hullin and Lafontaine [60] (Chapter 2, Proposition 2.61). Using Proposition 8.13 it is also possible to give a formula for $d\omega(X_0 \ldots, X_k)$ in terms of the ∇_{X_i}, where ω is any k-form, namely

$$d\omega(X_0 \ldots, X_k) = \sum_{i=0}^{k} (-1)^i \nabla_{X_i} \omega(X_1, \ldots, X_{i-1}, X_0, X_{i+1}, \ldots, X_k).$$

Again, the above formula in proved in Gallot, Hullin and Lafontaine [60] (Chapter 2, Proposition 2.61).

If ∇ is a metric connection, then we can say more about the parallel transport along a curve. Recall from Section 11.3, Definition 11.4, that a vector field, X, along a curve, γ, is parallel iff

$$\frac{DX}{dt} = 0.$$

The following proposition will be needed:

Proposition 11.24 *Given any Riemannian manifold, M, and any metric connection, ∇, on M, for every curve, $\gamma\colon [a,b] \to M$, on M, if X and Y are two vector fields along γ, then*

$$\frac{d}{dt}\langle X(t), Y(t)\rangle = \left\langle \frac{DX}{dt}, Y \right\rangle + \left\langle X, \frac{DY}{dt} \right\rangle.$$

Proof. (After John Milnor.) Using Proposition 11.13, we can pick some parallel vector fields, $Z_1, \ldots, Z_n$, along γ, such that $Z_1(a), \ldots, Z_n(a)$ form an orthogonal frame. Then, as in the proof of Proposition 11.12, in any chart, (U, φ), the vector fields X and Y along the portion of γ in U can be expressed as

$$X = \sum_{i=1}^{n} X_i(t) \frac{\partial}{\partial x_i}, \qquad Y = \sum_{i=1}^{n} Y_i(t) \frac{\partial}{\partial x_i},$$

and

$$\gamma'(t_0) = \sum_{i=1}^{n} \frac{du_i}{dt} \left(\frac{\partial}{\partial x_i}\right)_{\gamma(t_0)},$$

with $u_i = pr_i \circ \varphi \circ \gamma$. Let $\widetilde{X}$ and $\widetilde{Y}$ be two parallel vector fields along γ. As the vector fields, $\frac{\partial}{\partial x_i}$, can be extended over the whole space, M, as ∇ is a metric connection and as $\widetilde{X}$ and $\widetilde{Y}$ are parallel along γ, we get

$$d(\langle \widetilde{X}, \widetilde{Y}\rangle)(\gamma') = \gamma'[\langle \widetilde{X}, \widetilde{Y}\rangle] = \langle \nabla_{\gamma'} \widetilde{X}, \widetilde{Y}\rangle + \langle \widetilde{X}, \nabla_{\gamma'} \widetilde{Y}\rangle = 0.$$

So, $\langle \widetilde{X}, \widetilde{Y}\rangle$ is constant along the portion of γ in U. But then, $\langle \widetilde{X}, \widetilde{Y}\rangle$ is constant along γ. Applying this to the $Z_i(t)$, we see that $Z_1(t), \ldots, Z_n(t)$ is an orthogonal frame, for every $t \in [a,b]$. Then, we can write

$$X = \sum_{i=1} x_i Z_i, \qquad Y = \sum_{j=1} y_j Z_j,$$

where $x_i(t)$ and $y_i(t)$ are smooth real-valued functions. It follows that

$$\langle X(t), Y(t)\rangle = \sum_{i=1}^{n} x_i y_i$$

and that

$$\frac{DX}{dt} = \frac{dx_i}{dt} Z_i + x_i \frac{DZ_i}{dt} = \frac{dx_i}{dt} Z_i, \qquad \frac{DY}{dt} = \frac{dy_i}{dt} Z_i + y_i \frac{DZ_i}{dt} = \frac{dy_i}{dt} Z_i.$$

Therefore,

$$\left\langle \frac{DX}{dt}, Y \right\rangle + \left\langle X, \frac{DY}{dt} \right\rangle = \sum_{i=1}^{n} \left(\frac{dx_i}{dt}\, y_i + x_i\, \frac{dy_i}{dt} \right) = \frac{d}{dt}\langle X(t), Y(t)\rangle,$$

as claimed. $\square$

Using Proposition 11.24 we get

Proposition 11.25 *Given any Riemannian manifold, M, and any metric connection, ∇, on M, for every curve, $\gamma\colon [a,b] \to M$, on M, if X and Y are two vector fields along γ that are parallel, then*

$$\langle X, Y\rangle = C,$$

for some constant, C. In particular, $\|X(t)\|$ is constant. Furthermore, the linear isomorphism, $P_\gamma\colon T_{\gamma(a)} \to T_{\gamma(b)}$, is an isometry.

Proof. From Proposition 11.24, we have

$$\frac{d}{dt}\langle X(t), Y(t)\rangle = \left\langle \frac{DX}{dt}, Y\right\rangle + \left\langle X, \frac{DY}{dt}\right\rangle.$$

As X and Y are parallel along γ, we have $DX/dt = 0$ and $DY/dt = 0$, so

$$\frac{d}{dt}\langle X(t), Y(t)\rangle = 0,$$

which shows that $\langle X(t), Y(t)\rangle$ is constant. Therefore, for all $v, w \in T_{\gamma(a)}$, if X and Y are the unique vector fields parallel along γ such that $X(a) = v$ and $Y(a) = w$ given by Proposition 11.13, we have

$$\langle P_\gamma(v), P_\gamma(w)\rangle = \langle X(b), Y(b)\rangle = \langle X(a), Y(a)\rangle = \langle u, v\rangle,$$

which proves that P_γ is an isometry. $\square$

In particular, Proposition 11.25 shows that the holonomy group, $\mathrm{Hol}_p(\nabla)$, based at p, is a subgroup of $\mathbf{O}(n)$.

11.5 Duality between Vector Fields and Differential Forms and their Covariant Derivatives

If $(M, \langle -, -\rangle)$ is a Riemannian manifold, then the inner product, $\langle -, -\rangle_p$, on T_pM, establishes a canonical duality between T_pM and T_p^*M, as explained in Section 22.1. Namely, we have the isomorphism, $\flat\colon T_pM \to T_p^*M$, defined such that for every $u \in T_pM$, the linear form, $u^\flat \in T_p^*M$, is given by

$$u^\flat(v) = \langle u, v\rangle_p \qquad v \in T_pM.$$

The inverse isomorphism, $\sharp\colon T_p^*M \to T_pM$, is defined such that for every $\omega \in T_p^*M$, the vector, $\omega^\sharp$, is the unique vector in T_pM so that

$$\langle \omega^\sharp, v\rangle_p = \omega(v), \qquad v \in T_pM.$$

The isomorphisms $\flat$ and $\sharp$ induce isomorphisms between vector fields, $X \in \mathfrak{X}(M)$, and one-forms, $\omega \in \mathcal{A}^1(M)$: A vector field, $X \in \mathfrak{X}(M)$, yields the one-form, $X^\flat \in \mathcal{A}^1(M)$, given by

$$(X^\flat)_p = (X_p)^\flat$$

and a one-form, $\omega \in \mathcal{A}^1(M)$, yields the vector field, $\omega^\sharp \in \mathfrak{X}(M)$, given by

$$(\omega^\sharp)_p = (\omega_p)^\sharp,$$

so that

$$\omega_p(v) = \langle (\omega_p)^\sharp, v\rangle_p, \qquad v \in T_pM,\ p \in M.$$

In particular, for every smooth function, $f \in C^\infty(M)$, the vector field corresponding to the one-form, df, is the *gradient*, $\operatorname{grad} f$, of f. The gradient of f is uniquely determined by the condition

$$\langle (\operatorname{grad} f)_p, v\rangle_p = df_p(v), \qquad v \in T_pM,\ p \in M.$$

Recall from Proposition 11.5 that the covariant derivative, $\nabla_X\omega$, of any one-form, $\omega \in \mathcal{A}^1(M)$, is the one-form given by

$$(\nabla_X\omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y).$$

If ∇ is a metric connection, then the vector field, $(\nabla_X\omega)^\sharp$, corresponding to $\nabla_X\omega$ is nicely expressed in terms of $\omega^\sharp$: Indeed, we have

$$(\nabla_X\omega)^\sharp = \nabla_X\omega^\sharp.$$

The proof goes as follows:

$$\begin{aligned}
(\nabla_X\omega)(Y) &= X(\omega(Y)) - \omega(\nabla_X Y)\\
&= X(\langle \omega^\sharp, Y\rangle) - \langle \omega^\sharp, \nabla_X Y\rangle\\
&= \langle \nabla_X\omega^\sharp, Y\rangle + \langle \omega^\sharp, \nabla_X Y\rangle - \langle \omega^\sharp, \nabla_X Y\rangle\\
&= \langle \nabla_X\omega^\sharp, Y\rangle,
\end{aligned}$$

where we used the fact that the connection is compatible with the metric in the third line and so,

$$(\nabla_X\omega)^\sharp = \nabla_X\omega^\sharp,$$

as claimed.

11.6 Pontrjagin Classes and Chern Classes, a Glimpse

This section can be omitted at first reading. Its purpose is to introduce the reader to Pontrjagin Classes and Chern Classes which are fundamental invariants of real (resp. complex) vector bundles. We focus on motivations and intuitions and omit most proofs but we give precise pointers to the literature for proofs.

Given a real (resp. complex) rank n vector bundle, $\xi = (E, \pi, B, V)$, we know that locally, ξ "looks like" a trivial bundle, $U \times V$, for some open subset, U, of the base space, B, but globally, ξ can be very twisted and one of the main issues is to understand and quantify "how twisted" ξ really is. Now, we know that every vector bundle admit a connection, say ∇, and the curvature, R^{∇}, of this connection is some measure of the twisting of ξ. However, R^{∇} depends on ∇, so curvature is not intrinsic to ξ, which is unsatisfactory as we seek invariants that depend only on ξ.

Pontrjagin, Stiefel and Chern (starting from the late 1930's) discovered that invariants with "good" properties could be defined if we took these invariants to belong to various cohomology groups associated with B. Such invariants are usually called *characteristic classes*. Roughly, there are two main methods for defining characteristic classes, one using topology and the other, due to Chern and Weil, using differential forms. A masterly exposition of these methods is given in the classic book by Milnor and Stasheff [110]. Amazingly, the method of Chern and Weil using differential forms is quite accessible for someone who has reasonably good knowledge of differential forms and de Rham cohomology as long as one is willing to gloss over various technical details.

As we said earlier, one of the problems with curvature is that is depends on a connection. The way to circumvent this difficuty rests on the simple, yet subtle observation that locally, given any two overlapping local trivializations $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β), the transformation rule for the curvature matrices Ω_α and Ω_β is

$$\Omega_\beta = g_{\alpha\beta}\Omega_\alpha g_{\alpha\beta}^{-1},$$

where $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(V)$ is the transition function. The matrices of two-forms, Ω_α, are local, but the stroke of genius is to glue them together to form a global form using *invariant polynomials*.

Indeed, the Ω_α are $n \times n$ matrices so, consider the algebra of polynomials, $\mathbb{R}[X_1, \ldots, X_{n^2}]$ (or $\mathbb{C}[X_1, \ldots, X_{n^2}]$ in the complex case) in n^2 variables, considered as the entries of an $n \times n$ matrix. It is more convenient to use the set of variables $\{X_{ij} \mid 1 \leq i, j \leq n\}$, and to let X be the $n \times n$ matrix $X = (X_{ij})$.

Definition 11.7 A polynomial, $P \in \mathbb{R}[\{X_{ij} \mid 1 \leq i, j \leq n\}]$ (or $P \in \mathbb{C}[\{X_{ij} \mid 1 \leq i, j \leq n\}]$) is *invariant* iff

$$P(AXA^{-1}) = P(X),$$

for all $A \in \mathrm{GL}(n, \mathbb{R})$ (resp. $A \in \mathrm{GL}(n, \mathbb{C})$). The *algebra of invariant polynomials* over $n \times n$ matrices is denoted by I_n.

Examples of invariant polynomials are, the trace, $\mathrm{tr}(X)$, and the determinant, $\det(X)$, of the matrix X. We will characterize shortly the algebra I_n.

Now comes the punch line: For any homogeneous invariant polynomial, $P \in I_n$, of degree k, we can substitute Ω_α for X, that is, substitute ω_{ij} for X_{ij}, and evaluate $P(\Omega_\alpha)$. This is because Ω is a matrix of two-forms and the wedge product is commutative for forms of even degree. Therefore, $P(\Omega_\alpha) \in \mathcal{A}^{2k}(U_\alpha)$. But, the formula for a change of trivialization yields

$$P(\Omega_\alpha) = P(g_{\alpha\beta}\Omega_\alpha g_{\alpha\beta}^{-1}) = P(\Omega_\beta),$$

so the forms $P(\Omega_\alpha)$ and $P(\Omega_\beta)$ agree on overlaps and thus, they define a *global form* denoted $P(R^\nabla) \in \mathcal{A}^{2k}(B)$.

Now, we know how to obtain global $2k$-forms, $P(R^\nabla) \in \mathcal{A}^{2k}(B)$, but they still seem to depend on the connection and how do they define a cohomology class? Both problems are settled thanks to the following Theorems:

Theorem 11.26 *For every real rank n vector bundle, ξ, for every connection, ∇, on ξ, for every invariant homogeneous polynomial, P, of degree k, the $2k$-form, $P(R^\nabla) \in \mathcal{A}^{2k}(B)$, is closed. If ξ is a complex vector bundle, then the $2k$-form, $P(R^\nabla) \in \mathcal{A}^{2k}(B;\mathbb{C})$, is closed.*

Theorem 11.26 implies that the $2k$-form, $P(R^\nabla) \in \mathcal{A}^{2k}(B)$, defines a cohomology class, $[P(R^\nabla)] \in H^{2k}_{\mathrm{DR}}(B)$. We will come back to the proof of Theorem 11.26 later.

Theorem 11.27 *For every real (resp. complex) rank n vector bundle, for every invariant homogeneous polynomial, P, of degree k, the cohomology class, $[P(R^\nabla)] \in H^{2k}_{\mathrm{DR}}(B)$ (resp. $[P(R^\nabla)] \in H^{2k}_{\mathrm{DR}}(B;\mathbb{C})$) is independent of the choice of the connection ∇.*

The cohomology class, $[P(R^\nabla)]$, which does not depend on ∇ is denoted $P(\xi)$ and is called the *characteristic class* of ξ corresponding to P.

The proof of Theorem 11.27 involves a kind of homotopy argument, see Madsen and Tornehave [100] (Lemma 18.2), Morita [114] (Proposition 5.28) or see Milnor and Stasheff [110] (Appendix C).

The upshot is that Theorems 11.26 and 11.27 give us a method for producing invariants of a vector bundle that somehow reflect how curved (or twisted) the bundle is. However, it appears that we need to consider infinitely many invariants. Fortunately, we can do better because the algebra, I_n, of invariant polynomials is finitely generated and in fact, has very nice sets of generators. For this, we recall the *elementary symmetric functions* in n variables.

Given n variables, $\lambda_1, \ldots, \lambda_n$, we can write

$$\prod_{i=1}^{n}(1 + t\lambda_i) = 1 + \sigma_1 t + \sigma_2 t^2 + \cdots + \sigma_n t^n,$$

where the σ_i are symmetric, homogeneous polynomials of degree i in $\lambda_1, \ldots, \lambda_n$ called *elementary symmetric functions* in n variables. For example,

$$\sigma_1 = \sum_{i=1}^{n} \lambda_i, \quad \sigma_1 = \sum_{1 \le i < j \le n} \lambda_i\lambda_j, \quad \sigma_n = \lambda_1 \cdots \lambda_n.$$

To be more precise, we write $\sigma_i(\lambda_1, \ldots, \lambda_n)$ instead of σ_i.

Given any $n \times n$ matrix, $X = (X_{ij})$, we define $\sigma_i(X)$ by the formula

$$\det(I + tX) = 1 + \sigma_1(X)t + \sigma_2(X)t^2 + \cdots + \sigma_n(X)t^n.$$

We claim that

$$\sigma_i(X) = \sigma_i(\lambda_1, \ldots, \lambda_n),$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of X. Indeed, $\lambda_1, \ldots, \lambda_n$ are the roots the the polynomial $\det(\lambda I - X) = 0$, and as

$$\det(\lambda I - X) = \prod_{i=1}^{n}(\lambda - \lambda_i) = \lambda^n + \sum_{i=1}^{n}(-1)^i \sigma_i(\lambda_1, \ldots, \lambda_n)\lambda^{n-i},$$

by factoring λ^n and replacing λ^{-1} by $-\lambda^{-1}$, we get

$$\det(I + (-\lambda^{-1})X) = 1 + \sum_{i=1}^{n} \sigma_i(\lambda_1, \ldots, \lambda_n)(-\lambda^{-1})^n,$$

which proves our claim.

Observe that

$$\sigma_1(X) = \mathrm{tr}(X), \qquad \sigma_n(X) = \det(X).$$

Also, $\sigma_k(X^\top) = \sigma_k(X)$, since $\det(I + tX) = \det((I + tX)^\top) = \det(I + tX^\top)$. It is not very difficult to prove the following theorem:

Theorem 11.28 *The algebra, I_n, of invariant polynomials in n^2 variables is generated by $\sigma_1(X), \ldots, \sigma_n(X)$, that is*

$$I_n \cong \mathbb{R}[\sigma_1(X), \ldots, \sigma_n(X)] \qquad (\textit{resp.} \quad I_n \cong \mathbb{C}[\sigma_1(X), \ldots, \sigma_n(X)]).$$

For a proof of Theorem 11.28, see Milnor and Stasheff [110] (Appendix C, Lemma 6), Madsen and Tornehave [100] (Appendix B) or Morita [114] (Theorem 5.26). The proof uses the fact that for every matrix, X, there is an upper-triangular matrix, T, and an invertible matrix, B, so that

$$X = BTB^{-1}.$$

Then, we can replace B by the matrix $\mathrm{diag}(\epsilon, \epsilon^2, \ldots, \epsilon^n)B$, where ϵ is very small, and make the off diagonal entries arbitrarily small. By continuity, it follows that $P(X)$ depends only on the diagonal entries of BTB^{-1}, that is, on the eigenvalues of X. So, $P(X)$ must be a symmetric function of these eigenvalues and the classical theory of symmetric functions completes the proof.

It turns out that there are situations where it is more convenient to use another set of generators instead of $\sigma_1, \ldots, \sigma_n$. Define $s_i(X)$ by

$$s_i(X) = \mathrm{tr}(X^i).$$

Of course,

$$s_i(X) = \lambda_1^i + \cdots + \lambda_n^i,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of X. Now, the $\sigma_i(X)$ and $s_i(X)$ are related to each other by *Newton's formula*, namely:

$$s_i(X) - \sigma_1(X)s_{i-1}(X) + \sigma_2(X)s_{i-2}(X) + \cdots + (-1)^{i-1}\sigma_{i-1}(X)s_1(X) + (-1)^i i\sigma_i(X) = 0$$

with $1 \leq i \leq n$. A "cute" proof of the Newton formulae is obtained by computing the derivative of $\log(h(t))$, where

$$h(t) = \prod_{i=1}^{n}(1 + t\lambda_i) = 1 + \sigma_1 t + \sigma_2 t^2 + \cdots + \sigma_n t^n,$$

see Madsen and Tornehave [100] (Appendix B) or Morita [114] (Exercise 5.7).

Consequently, we can inductively compute s_i in terms of $\sigma_1, \ldots, \sigma_i$ and conversely, σ_i in terms of $s_1, \ldots, s_i$. For example,

$$s_1 = \sigma_1, \quad s_2 = \sigma_1^2 - 2\sigma_2, \quad s_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3.$$

It follows that

$$I_n \cong \mathbb{R}[s_1(X), \ldots, s_n(X)] \qquad (\text{resp.} \quad I_n \cong \mathbb{C}[s_1(X), \ldots, s_n(X)]).$$

Using the above, we can give a simple proof of Theorem 11.26, using Theorem 11.28.

Proof of Theorem 11.26. Since $s_1, \ldots, s_n$ generate I_n, it is enough to prove that $s_i(R^{\nabla})$ is closed. We need to prove that $ds_i(R^{\nabla}) = 0$ and for this, it is enough to prove it in every local trivialization, $(U_\alpha, \varphi_\alpha)$. To simplify notation, we write Ω for Ω_α. Now, $s_i(\Omega) = \mathrm{tr}(\Omega^i)$, so

$$ds_i(\Omega) = d\mathrm{tr}(\Omega^i) = \mathrm{tr}(d\Omega^i),$$

and we use Bianchi's identity (Proposition 11.10),

$$d\Omega = \omega \wedge \Omega - \Omega \wedge \omega.$$

We have

$$\begin{aligned}
d\Omega^i &= d\Omega \wedge \Omega^{i-1} + \Omega \wedge d\Omega \wedge \Omega^{i-2} + \cdots + \Omega^k \wedge d\Omega \wedge \Omega^{i-k-1} + \cdots + \Omega^{i-1} \wedge d\Omega \\
&= (\omega \wedge \Omega - \Omega \wedge \omega) \wedge \Omega^{i-1} + \Omega \wedge (\omega \wedge \Omega - \Omega \wedge \omega) \wedge \Omega^{i-2} \\
&\quad + \cdots + \Omega^k \wedge (\omega \wedge \Omega - \Omega \wedge \omega) \wedge \Omega^{i-k-1} + \Omega^{k+1} \wedge (\omega \wedge \Omega - \Omega \wedge \omega) \wedge \Omega^{i-k-2} \\
&\quad + \cdots + \Omega^{i-1} \wedge (\omega \wedge \Omega - \Omega \wedge \omega) \\
&= \omega \wedge \Omega^i - \Omega \wedge \omega \wedge \Omega^{i-1} + \Omega \wedge \omega \wedge \Omega^{i-1} - \Omega^2 \wedge \omega \wedge \Omega^{i-2} + \cdots + \\
&\quad \Omega^k \wedge \omega \wedge \Omega^{i-k} - \Omega^{k+1} \wedge \omega \wedge \Omega^{i-k-1} + \Omega^{k+1} \wedge \omega \wedge \Omega^{i-k-1} - \Omega^{k+2} \wedge \omega \wedge \Omega^{i-k-2} \\
&\quad + \cdots + \Omega^{i-1} \wedge \omega \wedge \Omega - \Omega^i \wedge \omega \\
&= \omega \wedge \Omega^i - \Omega^i \wedge \omega.
\end{aligned}$$

However, the entries in ω are one-forms, the entries in Ω are two-forms and since

$$\eta \wedge \theta = \theta \wedge \eta$$

for all $\eta \in \mathcal{A}^1(B)$ and all $\theta \in \mathcal{A}^2(B)$ and $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ for all matrices X and Y with commuting entries, we get

$$\mathrm{tr}(d\Omega^i) = \mathrm{tr}(\omega \wedge \Omega^i - \Omega^i \wedge \omega) = \mathrm{tr}(\omega \wedge \Omega^i) - \mathrm{tr}(\Omega^i \wedge \omega) = 0,$$

as required. $\square$

A more elegant proof (also using Bianchi's identity) can be found in Milnor and Stasheff [110] (Appendix C, page 296-298).

For real vector bundles, only invariant polynomials of even degrees matter.

Proposition 11.29 *If ξ is a real vector bundle, then for every homogeneous invariant polynomial, P, of odd degree, k, we have $P(\xi) = 0 \in H^{2k}_{\mathrm{DR}}(B)$.*

Proof. As $I_n \cong \mathbb{R}[s_1(X), \ldots, s_n(X)]$ and $s_i(X)$ is homogeneous of degree i, it is enough to prove Proposition 11.29 for $s_i(X)$ with i odd. By Theorem 11.27, we may assume that we pick a metric connection on ξ, so that Ω_α is skew-symmetric in every local trivialization. Then, Ω_α^i is also skew symmetric and

$$\mathrm{tr}(\Omega_\alpha^i) = 0,$$

since the diagonal entries of a real skew-symmetric matrix are all zero. It follows that $s_i(\Omega_\alpha) = \mathrm{tr}(\Omega_\alpha^i) = 0$. $\square$

Proposition 11.29 implies that for a real vector bundle, ξ, non-zero characteristic classes can only live in the cohomology groups $H^{4k}_{\mathrm{DR}}(B)$ of dimension $4k$. This property is specific to real vector bundles and generally fails for complex vector bundles.

Before defining Pontrjagin and Chern classes, we state another important properties of the homology classes, $P(\xi)$:

Proposition 11.30 *If $\xi = (E, \pi, B, V)$ and $\xi' = (E', \pi', B', V)$ are real (resp. complex) vector bundles, for every bundle map*

$$\begin{tikzcd} E \arrow[r, "f_E"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "f"'] & B', \end{tikzcd}$$

for every homogeneous invariant polynomial, P, of degree k, we have

$$P(\xi) = f^*(P(\xi')) \in H^{2k}_{\mathrm{DR}}(B) \qquad (\textit{resp.} \quad P(\xi) = f^*(P(\xi')) \in H^{2k}_{\mathrm{DR}}(B; \mathbb{C})).$$

In particular, for every smooth map, $f \colon N \to B$, we have

$$P(f^*\xi) = f^*(P(\xi)) \in H^{2k}_{\mathrm{DR}}(N) \qquad (\textit{resp.} \quad P(f^*\xi) = f^*(P(\xi)) \in H^{2k}_{\mathrm{DR}}(N; \mathbb{C})).$$

The above proposition implies that isomorphic vector bundles have identical characteristic classes. We finally define Pontrjagin classes and Chern classes.

Definition 11.8 If ξ be a real rank n vector bundle, then the k^{th} *Pontrjagin class* of ξ, denoted $p_k(\xi)$, where $1 \leq 2k \leq n$, is the cohomology class

$$p_k(\xi) = \left[\frac{1}{(2\pi)^{2k}}\,\sigma_{2k}(R^{\nabla})\right] \in H^{4k}_{\mathrm{DR}}(B),$$

for any connection, ∇, on ξ.

If ξ be a complex rank n vector bundle, then the k^{th} *Chern class* of ξ, denoted $c_k(\xi)$, where $1 \leq k \leq n$, is the cohomology class

$$c_k(\xi) = \left[\left(\frac{-1}{2\pi i}\right)^{k}\,\sigma_{k}(R^{\nabla})\right] \in H^{2k}_{\mathrm{DR}}(B),$$

for any connection, ∇, on ξ. We also set $p_0(\xi) = 1$ and $c_0(\xi) = 1$ in the complex case.

The strange coefficient in $p_k(\xi)$ is present so that our expression matches the topological definition of Pontrjagin classes. The equally strange coefficient in $c_k(\xi)$ is there to insure that $c_k(\xi)$ actually belongs to the *real* cohomology group $H^{2k}_{\mathrm{DR}}(B)$, as stated (from the definition we can only claim that $c_k(\xi) \in H^{2k}_{\mathrm{DR}}(B;\mathbb{C})$). This requires a proof which can be found in Morita [114] (Proposition 5.30) or in Madsen and Tornehave [100] (Chapter 18). One can use the fact that every complex vector bundle admits a Hermitian connection. Locally, the curvature matrices are skew-Hermitian and this easily implies that the Chern classes are real since if Ω is skew-Hermitian, then $i\Omega$ is Hermitian. (Actually, the topological version of Chern classes shows that $c_k(\xi) \in H^{2k}(B;\mathbb{Z})$.)

If ξ is a real rank n vector bundle and n is odd, say $n = 2m+1$, then the "top" Pontrjagin class, $p_m(\xi)$, corresponds to $\sigma_{2m}(R^{\nabla})$, which is not $\det(R^{\nabla})$. However, if n is even, say $n = 2m$, then the "top" Pontrjagin class $p_m(\xi)$ corresponds to $\det(R^{\nabla})$.

It is also useful to introduce the *Pontrjagin polynomial*, $p(\xi)(t) \in H^{\bullet}_{\mathrm{DR}}(B)[t]$, given by

$$p(\xi)(t) = \left[\det\left(I + \frac{t}{2\pi}\,R^{\nabla}\right)\right] = 1 + p_1(\xi)t + p_2(\xi)t^2 + \cdots + p_{\lfloor\frac{n}{2}\rfloor}(\xi)t^{\lfloor\frac{n}{2}\rfloor}$$

and the *Chern polynomial*, $c(\xi)(t) \in H^{\bullet}_{\mathrm{DR}}(B)[t]$, given by

$$c(\xi)(t) = \left[\det\left(I - \frac{t}{2\pi i}\,R^{\nabla}\right)\right] = 1 + c_1(\xi)t + c_2(\xi)t^2 + \cdots + c_n(\xi)t^n.$$

If a vector bundle is trivial, then all its Pontrjagin classes (or Chern classes) are zero for all $k \geq 1$. If ξ is the real tangent bundle, $\xi = TB$, of some manifold of dimension n, then the $\lfloor\frac{n}{4}\rfloor$ Pontrjagin classes of TB are denoted $p_1(B), \ldots, p_{\lfloor\frac{n}{4}\rfloor}(B)$.

For complex vector bundles, the manifold, B, is often the real manifold corresponding to a complex manifold. If B has complex dimension, n, then B has real dimension $2n$. In this case, the tangent bundle, TB, is a rank n complex vector bundle over the real manifold of dimension, $2n$, and thus, it has n Chern classes, denoted $c_1(B), \ldots, c_n(B)$. The determination of the Pontrjagin classes (or Chern classes) of a manifold is an important step for the study of the geometric/topological structure of the manifold. For example, it is possible to compute the Chern classes of complex projective space, $\mathbb{CP}^n$ (as a complex manifold).

The Pontrjagin classes of a real vector bundle, ξ, are related to the Chern classes of its complexification, $\xi_{\mathbb{C}} = \xi \otimes \epsilon^1_{\mathbb{C}}$ (where $\epsilon^1_{\mathbb{C}}$ is the trivial complex line bundle $B \times \mathbb{C}$).

Proposition 11.31 *For every real rank n vector bundle, $\xi = (E, \pi, B, V)$, if $\xi_{\mathbb{C}} = \xi \otimes \epsilon^1_{\mathbb{C}}$ is the complexification of ξ, then*

$$p_k(\xi) = (-1)^k c_{2k}(\xi_{\mathbb{C}}) \in H^{4k}_{\mathrm{DR}}(B) \qquad k \geq 0.$$

Basically, the reason why Proposition 11.31 holds is that

$$\frac{1}{(2\pi)^{2k}} = (-1)^k \left(\frac{-1}{2\pi i}\right)^{2k}$$

We conclude this section by stating a few more properties of Chern classes.

Proposition 11.32 *For every complex rank n vector bundle, ξ, the following properties hold:*

(1) If ξ has a Hermitian metric, then we have a canonical isomorphism, $\xi^* \cong \overline{\xi}$.

(2) The Chern classes of ξ, ξ^* and $\overline{\xi}$ satisfy:

$$c_k(\xi^*) = c_k(\overline{\xi}) = (-1)^k c_k(\xi).$$

(3) For any complex vector bundles, ξ and η,

$$c_k(\xi \oplus \eta) = \sum_{i=0}^{k} c_i(\xi) c_{k-i}(\eta)$$

or equivalently

$$c(\xi \oplus \eta)(t) = c(\xi)(t) c(\eta)(t)$$

and similarly for Pontrjagin classes when ξ and η are real vector bundles.

To prove (2) we can use the fact that ξ can be given a Hermitian metric. Then, we saw earlier that if ω is the connection matrix of ξ over U then $\overline{\omega} = -\omega^\top$ is the connection matrix of $\overline{\xi}$ over U. However, it is clear that $\sigma_k(-\Omega_\alpha^\top) = (-1)^k\sigma_k(\Omega_\alpha)$ and so, $c_k(\overline{\xi}) = (-1)^k c_k(\xi)$.

Remark: For a real vector bundle, ξ, it is easy to see that $(\xi_{\mathbb{C}})^* = (\xi^*)_{\mathbb{C}}$, which implies that $c_k((\xi_{\mathbb{C}})^*) = c_k(\xi_{\mathbb{C}})$ (as $\xi \cong \xi^*$) and (2) implies that $c_k(\xi_{\mathbb{C}}) = 0$ for k odd. This proves again that the Pontrjagin classes exit only in dimension $4k$.

A complex rank n vector bundle, ξ, can also be viewed as a rank $2n$ vector bundle, $\xi_{\mathbb{R}}$ and we have:

Proposition 11.33 *For every rank n complex vector bundle, ξ, if $p_k = p_k(\xi_{\mathbb{R}})$ and $c_k = c_k(\xi)$, then we have*

$$1 - p_1 + p_2 + \cdots + (-1)^n p_n = (1 + c_1 + c_2 + \cdots + c_n)(1 - c_1 + c_2 + \cdots + (-1)^n c_n).$$

11.7 Euler Classes and The Generalized Gauss-Bonnet Theorem

Let $\xi = (E, \pi, B, V)$ be a real vector bundle of rank $n = 2m$ and let ∇ be any metric connection on ξ. Then, if ξ is orientable (as defined in Section 7.4, see Definition 7.12 and the paragraph following it), it is possible to define a global form, $\mathrm{eu}(R^\nabla) \in \mathcal{A}^{2m}(B)$, which turns out to be closed. Furthermore, the cohomology class, $[\mathrm{eu}(R^\nabla)] \in H^{2m}_{\mathrm{DR}}(B)$, is independent of the choice of ∇. This cohomology class, denoted $e(\xi)$, is called the *Euler class* of ξ and has some very interesting properties. For example, $p_m(\xi) = e(\xi)^2$.

As ∇ is a metric connection, in a trivialization, $(U_\alpha, \varphi_\alpha)$, the curvature matrix, Ω_α, is a skew symmetric $2m \times 2m$ matrix of 2-forms. Therefore, we can substitute the 2-forms in Ω_α for the variables of the Pfaffian of degree m (see Section 22.20) and we obtain the $2m$-form, $\mathrm{Pf}(\Omega_\alpha) \in \mathcal{A}^{2m}(B)$. Now, as ξ is orientable, the transition functions take values in $\mathbf{SO}(2m)$, so by Proposition 11.9, since

$$\Omega_\beta = g_{\alpha\beta}\Omega_\alpha g_{\alpha\beta}^{-1},$$

we conclude from Proposition 22.38 (ii) that

$$\mathrm{Pf}(\Omega_\alpha) = \mathrm{Pf}(\Omega_\beta).$$

Therefore, the local $2m$-forms, $\mathrm{Pf}(\Omega_\alpha)$, patch and define a global form, $\mathrm{Pf}(R^\nabla) \in \mathcal{A}^{2m}(B)$.

The following propositions can be shown:

Proposition 11.34 *For every real, orientable, rank $2m$ vector bundle, ξ, for every metric connection, ∇, on ξ the $2m$-form, $\mathrm{Pf}(R^\nabla) \in \mathcal{A}^{2m}(B)$, is closed.*

Proposition 11.35 *For every real, orientable, rank $2m$ vector bundle, ξ, the cohomology class, $[\mathrm{Pf}(R^{\nabla})] \in H^{2m}_{\mathrm{DR}}(B)$, is independent of the metric connection, ∇, on ξ.*

Proofs of Propositions 11.34 and 11.35 can be found in Madsen and Tornehave [100] (Chapter 19) or Milnor and Stasheff [110] (Appendix C) (also see Morita [114], Chapters 5 and 6).

Definition 11.9 Let $\xi = (E, \pi, B, V)$ be any real, orientable, rank $2m$ vector bundle. For any metric connection, ∇, on ξ the *Euler form* associated with ∇ is the closed form

$$\mathrm{eu}(R^{\nabla}) = \frac{1}{(2\pi)^n}\,\mathrm{Pf}(R^{\nabla}) \in \mathcal{A}^{2m}(B)$$

and the *Euler class* of ξ is the cohomology class,

$$e(\xi) = \big[\mathrm{eu}(R^{\nabla})\big] \in H^{2m}_{\mathrm{DR}}(B),$$

which does not depend on ∇.

Some authors, including Madsen and Tornehave [100], have a negative sign in front of R^{∇} in their definition of the Euler form, that is, they define $\mathrm{eu}(R^{\nabla})$ by

$$\mathrm{eu}(R^{\nabla}) = \frac{1}{(2\pi)^n}\,\mathrm{Pf}(-R^{\nabla}).$$

However these authors use a Pfaffian with the opposite sign convention from ours and this Pfaffian differs from ours by the factor $(-1)^n$ (see the warning in Section 22.20). Madsen and Tornehave [100] seem to have overlooked this point and with their definition of the Pfaffian (which is the one we have adopted) Proposition 11.37 is incorrect.

Here is the relationship between the Euler class, $e(\xi)$, and the top Pontrjagin class, $p_m(\xi)$:

Proposition 11.36 *For every real, orientable, rank $2m$ vector bundle, $\xi = (E, \pi, B, V)$, we have*

$$p_m(\xi) = e(\xi)^2 \in H^{4m}_{\mathrm{DR}}(B).$$

Proof. The top Pontrjagin class, $p_m(\xi)$, is given by

$$p_m(\xi) = \left[\frac{1}{(2\pi)^{2m}}\,\det(R^{\nabla})\right],$$

for any (metric) connection, ∇ and

$$e(\xi) = \big[\mathrm{eu}(R^{\nabla})\big]$$

with

$$\mathrm{eu}(R^\nabla) = \frac{1}{(2\pi)^n}\,\mathrm{Pf}(R^\nabla).$$

From Proposition 22.38 (i), we have

$$\det(R^\nabla) = \mathrm{Pf}(R^\nabla)^2,$$

which yields the desired result. $\square$

A rank m complex vector bundle, $\xi = (E, \pi, B, V)$, can be viewed as a real rank $2m$ vector bundle, $\xi_{\mathbb{R}}$, by viewing V as a $2m$ dimensional real vector space. Then, it turns out that $\xi_{\mathbb{R}}$ is naturally orientable. Here is the reason.

For any basis, $(e_1, \ldots, e_m)$, of V over $\mathbb{C}$, observe that $(e_1, ie_1, \ldots, e_m, ie_m)$ is a basis of V over $\mathbb{R}$ (since $v = \sum_{i=1}^m (\lambda_i + i\mu_i)e_i = \sum_{i=1}^m \lambda_i e_i + \sum_{i=1}^m \mu_i i e_i$). But, any $m \times m$ invertible matrix, A, over $\mathbb{C}$ becomes a real $2m \times 2m$ invertible matrix, $A_{\mathbb{R}}$, obtained by replacing the entry $a_{jk} + ib_{jk}$ in A by the real 2×2 matrix

$$\begin{pmatrix} a_{jk} & -b_{jk} \\ b_{jk} & a_{jk}. \end{pmatrix}$$

Indeed, if $v_k = \sum_{j=1}^m a_{jk}e_j + \sum_{j=1}^m b_{jk} i e_j$, then $iv_k = \sum_{j=1}^m -b_{jk}e_j + \sum_{j=1}^m a_{jk} i e_j$ and when we express v_k and iv_k over the basis $(e_1, ie_1, \ldots, e_m, ie_m)$, we get a matrix $A_{\mathbb{R}}$ consisting of 2×2 blocks as above. Clearly, the map $r\colon A \mapsto A_{\mathbb{R}}$ is a continuous injective homomorphism from $\mathrm{GL}(m, \mathbb{C})$ to $\mathrm{GL}(2m, \mathbb{R})$. Now, it is known $\mathrm{GL}(m, \mathbb{C})$ is connected, thus $\mathrm{Im}(r) = r(\mathrm{GL}(m, \mathbb{C}))$ is connected and as $\det(I_{2m}) = 1$, we conclude that all matrices in $\mathrm{Im}(r)$ have positive determinant.[1] Therefore, the transition functions of $\xi_{\mathbb{R}}$ which take values in $\mathrm{Im}(r)$ have positive determinant and $\xi_{\mathbb{R}}$ is orientable. We can give $\xi_{\mathbb{R}}$ an orientation by fixing some basis of V over $\mathbb{R}$. Then, we have the following relationship between $e(\xi_{\mathbb{R}})$ and the top Chern class, $c_m(\xi)$:

Proposition 11.37 *For every complex, rank m vector bundle, $\xi = (E, \pi, B, V)$, we have*

$$c_m(\xi) = e(\xi) \in H^{2m}_{\mathrm{DR}}(B).$$

Proof. Pick some metric connection, ∇. Recall that

$$c_m(\xi) = \left[\left(\frac{-1}{2\pi i}\right)^m \det(R^\nabla)\right] = i^m \left[\left(\frac{1}{2\pi}\right)^m \det(R^\nabla)\right].$$

On the other hand,

$$e(\xi) = \left[\frac{1}{(2\pi)^m}\,\mathrm{Pf}(R^\nabla_{\mathbb{R}})\right].$$

[1] One can also prove directly that every matrix in $\mathrm{Im}(r)$ has positive determinant by expressing $r(A)$ as a product of simple matrices whose determinants are easily computed.

Here, $R^{\nabla}_{\mathbb{R}}$ denotes the global $2m$-form wich, locally, is equal to $\Omega_{\mathbb{R}}$, where Ω is the $m \times m$ curvature matrix of ξ over some trivialization. By Proposition 22.39,

$$\mathrm{Pf}(\Omega_{\mathbb{R}}) = i^n \det(\Omega),$$

so $c_m(\xi) = e(\xi)$, as claimed. $\square$

The Euler class enjoys many other nice properties. For example, if $f\colon \xi_1 \to \xi_2$ is an orientation preserving bundle map, then

$$e(f^*\xi_2) = f^*(e(\xi_2)),$$

where $f^*\xi_2$ is given the orientation induced by ξ_2. Also, the Euler class can be defined by topological means and it belongs to the integral cohomology group $H^{2m}(B; \mathbb{Z})$.

Although this result lies beyond the scope of these notes we cannot resist stating one of the most important and most beautiful theorems of differential geometry usually called the *Generalized Gauss-Bonnet Theorem* or *Gauss-Bonnet-Chern Theorem*.

For this, we need the notion of Euler characteristic. Since we haven't discussed triangulations of manifolds, we will use a defintion in terms of cohomology. Although concise, this definition is hard to motivate and we appologize for this. Given a smooth n-dimensional manifold, M, we define its *Euler characteristic*, $\chi(M)$, as

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim(H^i_{\mathrm{DR}}).$$

The integers, $b_i = \dim(H^i_{\mathrm{DR}})$, are known as the *Betti numbers* of M. For example, $\chi(S^2) = 2$.

It turns out that if M is an odd dimensional manifold, then $\chi(M) = 0$. This explains partially why the Euler class is only defined for even dimensional bundles.

The Generalized Gauss-Bonnet Theorem (or Gauss-Bonnet-Chern Theorem) is a generalization of the Gauss-Bonnet Theorem for surfaces. In the general form stated below it was first proved by Allendoerfer and Weil (1943), and Chern (1944).

Theorem 11.38 *(Generalized Gauss-Bonnet Formula) Let M be an orientable, smooth, compact manifold of dimension $2m$. For every metric connection, ∇, on TM, (in particular, the Levi-Civita connection for a Riemannian manifold) we have*

$$\int_M \mathrm{eu}(R^{\nabla}) = \chi(M).$$

A proof of Theorem 11.38 can be found in Madsen and Tornehave [100] (Chapter 21), but beware of some sign problems. The proof uses another famous theorem of differential topology, the *Poincaré-Hopf Theorem*. A sketch of the proof is also given in Morita [114], Chapter 5.

Theorem 11.38 is remarkable because it establishes a relationship between the geometry of the manifold (its curvature) and the topology of the manifold (the number of “holes”), somehow encoded in its Euler characteristic.

Characteristic classes are a rich and important topic and we’ve only scratched the surface. We refer the reader to the texts mentioned earlier in this section as well as to Bott and Tu [19] for comprehensive expositions.

Chapter 12

Geodesics on Riemannian Manifolds

12.1 Geodesics, Local Existence and Uniqueness

If (M, g) is a Riemannian manifold, then the concept of *length* makes sense for any piecewise smooth (in fact, C^1) curve on M. Then, it possible to define the structure of a metric space on M, where $d(p, q)$ is the greatest lower bound of the length of all curves joining p and q. Curves on M which locally yield the shortest distance between two points are of great interest. These curves called *geodesics* play an important role and the goal of this chapter is to study some of their properties. Since geodesics are a standard chapter of every differential geometry text, we will omit most proofs and instead give precise pointers to the literature. Among the many presentations of this subject, in our opinion, Milnor's account [106] (Part II, Section 11) is still one of the best, certainly by its clarity and elegance. We acknowledge that our presentation was heavily inspired by this beautiful work. We also relied heavily on Gallot, Hulin and Lafontaine [60] (Chapter 2), Do Carmo [50], O'Neill [119], Kuhnel [91] and class notes by Pierre Pansu (see http://www.math.u-psud.fr/%7Epansu/web_dea/resume_dea_04.html in http://www.math.u-psud.fr˜pansu/). Another reference that is remarkable by its clarity and the completeness of its coverage is Postnikov [125].

Given any $p \in M$, for every $v \in T_pM$, the *(Riemannian) norm* of v, denoted $\|v\|$, is defined by

$$\|v\| = \sqrt{g_p(v, v)}.$$

The Riemannian inner product, $g_p(u, v)$, of two tangent vectors, $u, v \in T_pM$, will also be denoted by $\langle u, v\rangle_p$, or simply $\langle u, v\rangle$. Recall the following definitions regarding curves:

Definition 12.1 Given any Riemannian manifold, M, a *smooth parametric curve* (for short, *curve*) on M is a map, $\gamma\colon I \to M$, where I is some open interval of $\mathbb{R}$. For a closed interval, $[a, b] \subseteq \mathbb{R}$, a map $\gamma\colon [a, b] \to M$ is a *smooth curve from $p = \gamma(a)$ to $q = \gamma(b)$* iff γ can be extended to a smooth curve $\widetilde{\gamma}\colon (a - \epsilon, b + \epsilon) \to M$, for some $\epsilon > 0$. Given any two points, $p, q \in M$, a continuous map, $\gamma\colon [a, b] \to M$, is a *piecewise smooth curve from p to q* iff

(1) There is a sequence $a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b$ of numbers, $t_i \in \mathbb{R}$, so that each map, $\gamma \restriction [t_i, t_{i+1}]$, called a *curve segment* is a smooth curve, for $i = 0, \ldots, k-1$.

(2) $\gamma(a) = p$ and $\gamma(b) = q$.

The set of all piecewise smooth curves from p to q is denoted $\Omega(M; p, q)$, or briefly $\Omega(p, q)$ (or even Ω, when p and q are understood).

The set $\Omega(M; p, q)$ is an important object sometimes called the *path space* of M (from p to q). Unfortunately it is an infinite-dimensional manifold, which makes it hard to investigate its properties.

Observe that at any junction point, $\gamma_{i-1}(t_i) = \gamma_i(t_i)$, there may be a jump in the velocity vector of γ. We let $\gamma'((t_i)_+) = \gamma'_{i-1}(t_i)$ and $\gamma'((t_i)_-) = \gamma'_i(t_i)$.

Given any curve, $\gamma \in \Omega(M; p, q)$, the *length*, $L(\gamma)$, of γ is defined by

$$L(\gamma) = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \|\gamma'(t)\| \, dt = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \sqrt{g(\gamma'(t), \gamma'(t))} \, dt.$$

It is easy to see that $L(\gamma)$ is unchanged by a monotone reparametrization (that is, a map $h \colon [a, b] \to [c, d]$, whose derivative, h', has a constant sign).

Let us now assume that our Riemannian manifold, (M, g), is equipped with the Levi-Civita connection and thus, for every curve, γ, on M, let $\frac{D}{dt}$ be the associated covariant derivative along γ, also denoted $\nabla_{\gamma'}$

Definition 12.2 Let (M, g) be a Riemannian manifold. A curve, $\gamma \colon I \to M$, (where $I \subseteq \mathbb{R}$ is any interval) is a *geodesic* iff $\gamma'(t)$ is parallel along γ, that is, iff

$$\frac{D\gamma'}{dt} = \nabla_{\gamma'}\gamma' = 0.$$

If M was embedded in $\mathbb{R}^d$, a geodesic would be a curve, γ, such that the acceleration vector, $\gamma'' = \frac{D\gamma'}{dt}$, is normal to $T_{\gamma(t)}M$.

By Proposition 11.25, $\|\gamma'(t)\| = \sqrt{g(\gamma'(t), \gamma'(t))}$ is constant, say $\|\gamma'(t)\| = c$. If we define the *arc-length* function, $s(t)$, relative to a, where a is any chosen point in I, by

$$s(t) = \int_a^t \sqrt{g(\gamma'(t), \gamma'(t))} \, dt = c(t - a), \qquad t \in I,$$

we conclude that for a geodesic, $\gamma(t)$, the parameter, t, is an affine function of the arc-length. When $c = 1$, which can be achieved by an affine reparametrization, we say that the geodesic is *normalized*.

The geodesics in $\mathbb{R}^n$ are the straight lines parametrized by constant velocity. The geodesics of the 2-sphere are the great circles, parametrized by arc-length. The geodesics of the Poincaré half-plane are the lines $x = a$ and the half-circles centered on the x-axis. The geodesics of an ellipsoid are quite fascinating. They can be completely characterized and they are parametrized by elliptic functions (see Hilbert and Cohn-Vossen [75], Chapter 4, Section and Berger and Gostiaux [17], Section 10.4.9.5). If M is a submanifold of $\mathbb{R}^n$, geodesics are curves whose acceleration vector, $\gamma'' = (D\gamma')/dt$ is normal to M (that is, for every $p \in M$, γ'' is normal to T_pM).

In a local chart, (U, φ), since a geodesic is characterized by the fact that its velocity vector field, $\gamma'(t)$, along γ is parallel, by Proposition 11.13, it is the solution of the following system of second-order ODE's in the unknowns, u_k:

$$\frac{d^2u_k}{dt^2} + \sum_{ij} \Gamma^k_{ij} \frac{du_i}{dt}\frac{du_j}{dt} = 0, \qquad k = 1, \ldots, n,$$

with $u_i = pr_i \circ \varphi \circ \gamma$ ($n = \dim(M)$).

The standard existence and uniqueness results for ODE's can be used to prove the following proposition (see O'Neill [119], Chapter 3):

Proposition 12.1 *Let (M, g) be a Riemannian manifold. For every point, $p \in M$, and every tangent vector, $v \in T_pM$, there is some interval, $(-\eta, \eta)$, and a unique geodesic,*

$$\gamma_v \colon (-\eta, \eta) \to M,$$

satisfying the conditions

$$\gamma_v(0) = p, \qquad \gamma_v'(0) = v.$$

The following proposition is used to prove that every geodesic is contained in a unique maximal geodesic (*i.e*, with largest possible domain). For a proof, see O'Neill [119], Chapter 3 or Petersen [121] (Chapter 5, Section 2, Lemma 7).

Proposition 12.2 *For any two geodesics, $\gamma_1 \colon I_1 \to M$ and $\gamma_2 \colon I_2 \to M$, if $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1'(a) = \gamma_2'(a)$, for some $a \in I_1 \cap I_2$, then $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$.*

Propositions 12.1 and 12.2 imply that for every $p \in M$ and every $v \in T_pM$, there is a unique geodesic, denoted γ_v, such that $\gamma(0) = p$, $\gamma'(0) = v$, and the domain of γ is the largest possible, that is, cannot be extended. We call γ_v a *maximal geodesic* (with initial conditions $\gamma_v(0) = p$ and $\gamma_v'(0) = v$).

Observe that the system of differential equations satisfied by geodesics has the following homogeneity property: If $t \mapsto \gamma(t)$ is a solution of the above system, then for every constant, c, the curve $t \mapsto \gamma(ct)$ is also a solution of the system. We can use this fact together with standard existence and uniqueness results for ODE's to prove the proposition below. For proofs, see Milnor [106] (Part II, Section 10), or Gallot, Hulin and Lafontaine [60] (Chapter 2).

Proposition 12.3 *Let (M, g) be a Riemannian manifold. For every point, $p_0 \in M$, there is an open subset, $U \subseteq M$, with $p_0 \in U$, and some $\epsilon > 0$, so that: For every $p \in U$ and every tangent vector, $v \in T_pM$, with $\|v\| < \epsilon$, there is a unique geodesic,*

$$\gamma_v \colon (-2, 2) \to M,$$

satisfying the conditions

$$\gamma_v(0) = p, \qquad \gamma_v'(0) = v.$$

If $\gamma_v \colon (-\eta, \eta) \to M$ is a geodesic with initial conditions $\gamma_v(0) = p$ and $\gamma_v'(0) = v \neq 0$, for any constant, $c \neq 0$, the curve, $t \mapsto \gamma_v(ct)$, is a geodesic defined on $(-\eta/c, \eta/c)$ (or $(\eta/c, -\eta/c)$ if $c < 0$) such that $\gamma'(0) = cv$. Thus,

$$\gamma_v(ct) = \gamma_{cv}(t), \qquad ct \in (-\eta, \eta).$$

This fact will be used in the next section.

Given any function, $f \in C^\infty(M)$, for any $p \in M$ and for any $u \in T_pM$, the value of the Hessian, $\mathrm{Hess}_p(f)(u, u)$, can be computed using geodesics. Indeed, for any geodesic, $\gamma \colon [0, \epsilon] \to M$, such that $\gamma(0) = p$ and $\gamma'(0) = u$, we have

$$\mathrm{Hess}_p(u, u) = \gamma'(\gamma'(f)) - (\nabla_{\gamma'}\gamma')(f) = \gamma'(\gamma'(f))$$

since $\nabla_{\gamma'}\gamma' = 0$ because γ is a geodesic and

$$\gamma'(\gamma'(f)) = \gamma'(df(\gamma')) = \gamma'\left(\frac{d}{dt} f(\gamma(t))\bigg|_{t=0}\right) = \frac{d^2}{dt^2} f(\gamma(t))\bigg|_{t=0},$$

and thus,

$$\mathrm{Hess}_p(u, u) = \frac{d^2}{dt^2} f(\gamma(t))\bigg|_{t=0}.$$

12.2 The Exponential Map

The idea behind the exponential map is to parametrize a Riemannian manifold, M, locally near any $p \in M$ in terms of a map from the tangent space T_pM to the manifold, this map being defined in terms of geodesics.

Definition 12.3 Let (M, g) be a Riemannian manifold. For every $p \in M$, let $\mathcal{D}(p)$ (or simply, $\mathcal{D}$) be the open subset of T_pM given by

$$\mathcal{D}(p) = \{v \in T_pM \mid \gamma_v(1) \quad \text{is defined}\},$$

where γ_v is the unique maximal geodesic with initial conditions $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. The *exponential map* is the map, $\exp_p \colon \mathcal{D}(p) \to M$, given by

$$\exp_p(v) = \gamma_v(1).$$

It is easy to see that $\mathcal{D}(p)$ is *star-shaped*, which means that if $w \in \mathcal{D}(p)$, then the line segment $\{tw \mid 0 \leq t \leq 1\}$ is contained in $\mathcal{D}(p)$. In view of the remark made at the end of the previous section, the curve

$$t \mapsto \exp_p(tv), \qquad tv \in \mathcal{D}(p)$$

is the geodesic, γ_v, through p such that $\gamma_v'(0) = v$. Such geodesics are called *radial geodesics*. The point, $\exp_p(tv)$, is obtained by running along the geodesic, γ_v, an arc length equal to $t\|v\|$, starting from p.

In general, $\mathcal{D}(p)$ is a proper subset of T_pM. For example, if U is a bounded open subset of $\mathbb{R}^n$, since we can identify T_pU with $\mathbb{R}^n$ for all $p \in U$, then $\mathcal{D}(p) \subseteq U$, for all $p \in U$.

Definition 12.4 A Riemannian manifold, (M, g), is *geodesically complete* iff $\mathcal{D}(p) = T_pM$, for all $p \in M$, that is, iff the exponential, $\exp_p(v)$, is defined for all $p \in M$ and for all $v \in T_pM$.

Equivalently, (M, g) is geodesically complete iff every geodesic can be extended indefinitely. Geodesically complete manifolds have nice properties, some of which will be investigated later.

Observe that $d(\exp_p)_0 = \mathrm{id}_{T_pM}$. This is because, for every $v \in \mathcal{D}(p)$, the map $t \mapsto \exp_p(tv)$ is the geodesic, γ_v, and

$$\frac{d}{dt}(\gamma_v(t))|_{t=0} = v = \frac{d}{dt}(\exp_p(tv))|_{t=0} = d(\exp_p)_0(v).$$

It follows from the inverse function theorem that $\exp_p$ is a diffeomorphism from some open ball in T_pM centered at 0 to M. The following slightly stronger proposition can be shown (Milnor [106], Chapter 10, Lemma 10.3):

Proposition 12.4 *Let (M, g) be a Riemannian manifold. For every point, $p \in M$, there is an open subset, $W \subseteq M$, with $p \in W$ and a number $\epsilon > 0$, so that*

(1) Any two points q_1, q_2 of W are joined by a unique geodesic of length $< \epsilon$.

(2) This geodesic depends smoothly upon q_1 and q_2, that is, if $t \mapsto \exp_{q_1}(tv)$ is the geodesic joining q_1 and q_2 $(0 \leq t \leq 1)$, then $v \in T_{q_1}M$ depends smoothly on (q_1, q_2).

(3) For every $q \in W$, the map $\exp_q$ is a diffeomorphism from the open ball, $B(0, \epsilon) \subseteq T_qM$, to its image, $U_q = \exp_q(B(0, \epsilon)) \subseteq M$, with $W \subseteq U_q$ and U_q open.

For any $q \in M$, an open neighborhood of q of the form, $U_q = \exp_q(B(0, \epsilon))$, where $\exp_q$ is a diffeomorphism from the open ball $B(0, \epsilon)$ onto U_q, is called a *normal neighborhood*.

Definition 12.5 Let (M, g) be a Riemannian manifold. For every point, $p \in M$, the *injectivity radius of M at p*, denoted $i(p)$, is the least upper bound of the numbers, $r > 0$, such that $\exp_p$ is a diffeomorphism on the open ball $B(0, r) \subseteq T_pM$. The *injectivity radius, $i(M)$, of M* is the greatest lower bound of the numbers, $i(p)$, where $p \in M$.

For every $p \in M$, we get a chart, (U_p, φ), where $U_p = \exp_p(B(0, i(p)))$ and $\varphi = \exp^{-1}$, called a *normal chart*. If we pick any orthonormal basis, $(e_1, \ldots, e_n)$, of T_pM, then the x_i's, with $x_i = pr_i \circ \exp^{-1}$ and pr_i the projection onto $\mathbb{R}e_i$, are called *normal coordinates* at p (here, $n = \dim(M)$). These are defined up to an isometry of T_pM. The following proposition shows that Riemannian metrics do not admit any local invariants of order one. The proof is left as an exercise.

Proposition 12.5 *Let (M, g) be a Riemannian manifold. For every point, $p \in M$, in normal coordinates at p,*

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)_p = \delta_{ij} \qquad \textit{and} \qquad \Gamma^k_{ij}(p) = 0.$$

For the next proposition, known as *Gauss Lemma*, we need to define *polar coordinates* on T_pM. If $n = \dim(M)$, observe that the map, $(0, \infty) \times S^{n-1} \longrightarrow T_pM - \{0\}$, given by

$$(r, v) \mapsto rv, \qquad r > 0,\ v \in S^{n-1}$$

is a diffeomorphism, where S^{n-1} is the sphere of radius $r = 1$ in T_pM. Then, the map, $f \colon (0, i(p)) \times S^{n-1} \to U_p - \{p\}$, given by

$$(r, v) \mapsto \exp_p(rv), \qquad 0 < r < i(p),\ v \in S^{n-1}$$

is also a diffeomorphism.

Proposition 12.6 *(Gauss Lemma) Let (M, g) be a Riemannian manifold. For every point, $p \in M$, the images, $\exp_p(S(0, r))$, of the spheres, $S(0, r) \subseteq T_pM$, centered at 0 by the exponential map, $\exp_p$, are orthogonal to the radial geodesics, $r \mapsto \exp_p(rv)$, through p, for all $r < i(p)$. Furthermore, in polar coordinates, the pull-back metric, $\exp^* g$, induced on T_pM is of the form*

$$exp^* g = dr^2 + g_r,$$

where g_r is a metric on the unit sphere, S^{n-1}, with the property that g_r/r^2 converges to the standard metric on S^{n-1} (induced by $\mathbb{R}^n$) when r goes to zero (here, $n = \dim(M)$).

Sketch of proof. (After Milnor, see [106], Chapter II, Section 10.) Pick any curve, $t \mapsto v(t)$ on the unit sphere, S^{n-1}. We must show that the corresponding curve on M,

$$t \mapsto \exp_p(rv(t)),$$

with r fixed, is orthogonal to the radial geodesic,

$$r \mapsto \exp_p(rv(t)),$$

with t fixed, $0 \leq r < i(p)$. In terms of the parametrized surface,

$$f(r,t) = \exp_p(rv(t)),$$

we must prove that

$$\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle = 0,$$

for all (r,t). However, as we are using the Levi-Civita connection which is compatible with the metric, we have

$$\frac{\partial}{\partial r}\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle = \left\langle \frac{D}{\partial r}\frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle + \left\langle \frac{\partial f}{\partial r}, \frac{D}{\partial r}\frac{\partial f}{\partial t} \right\rangle.$$

The first expression on the right is zero since the curves

$$t \mapsto f(r,t)$$

are geodesics. For the second expression, we have

$$\left\langle \frac{\partial f}{\partial r}, \frac{D}{\partial r}\frac{\partial f}{\partial t} \right\rangle = \frac{1}{2}\frac{\partial}{\partial t}\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial r} \right\rangle = 0,$$

since $1 = \|v(t)\| = \|\partial f/\partial r\|$. Therefore,

$$\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle$$

is independent of r. But, for $r = 0$, we have

$$f(0,t) = \exp_p(0) = p,$$

hence

$$\partial f/\partial t(0,t) = 0$$

and thus,

$$\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle = 0$$

for all r, t, which concludes the proof of the first statement. For the proof of the second statement, see Pansu's class notes, Chapter 3, Section 3.5. $\square$

Consider any piecewise smooth curve

$$\omega\colon [a,b] \to U_p - \{p\}.$$

We can write each point $\omega(t)$ uniquely as

$$\omega(t) = \exp_p(r(t)v(t)),$$

with $0 < r(t) < i(p)$, $v(t) \in T_pM$ and $\|v(t)\| = 1$.

Proposition 12.7 *Let (M, g) be a Riemannian manifold. We have*

$$\int_a^b \|\omega'(t)\| \; dt \geq |r(b) - r(a)|,$$

where equality holds only if the function r is monotone and the function v is constant. Thus, the shortest path joining two concentric spherical shells, $\exp_p(S(0, r_1))$ and $\exp_p(S(0, r_2))$, is a radial geodesic.

Proof. (After Milnor, see [106], Chapter II, Section 10.) Again, let $f(r, t) = \exp_p(rv(t))$, so that $\omega(t) = f(r(t), t)$. Then,

$$\frac{d\omega}{dt} = \frac{\partial f}{\partial r} r'(t) + \frac{\partial f}{\partial t}.$$

The proof of the previous proposition showed that the two vectors on the right-hand side are orthogonal and since $\|\partial f/\partial r\| = 1$, this gives

$$\left\| \frac{d\omega}{dt} \right\|^2 = |r'(t)|^2 + \left\| \frac{\partial f}{\partial t} \right\|^2 \geq |r'(t)|^2$$

where equality holds only if $\partial f/\partial t = 0$; hence only if $v'(t) = 0$. Thus,

$$\int_a^b \left\| \frac{d\omega}{dt} \right\| \; dt \geq \int_a^b |r'(t)| dt \geq |r(b) - r(a)|$$

where equality holds only if $r(t)$ is monotone and $v(t)$ is constant. $\square$

We now get the following important result from Proposition 12.6 and Proposition 12.7:

Theorem 12.8 *Let (M, g) be a Riemannian manifold. Let W and ϵ be as in Proposition 12.4 and let $\gamma\colon [0, 1] \to M$ be the geodesic of length $< \epsilon$ joining two points q_1, q_2 of W. For any other piecewise smooth path, ω, joining q_1 and q_2, we have*

$$\int_0^1 \|\gamma'(t)\| \, dt \leq \int_0^1 \|\omega'(t)\| \, dt$$

where equality can holds only if the images $\omega([0, 1])$ and $\gamma([0, 1])$ coincide. Thus, γ is the shortest path from q_1 to q_2.

Proof. (After Milnor, see [106], Chapter II, Section 10.) Consider any piecewise smooth path, ω, from $q_1 = \gamma(0)$ to some point

$$q_2 = \exp_{q_1}(rv) \in U_{q_1},$$

where $0 < r < \epsilon$ and $\|v\| = 1$. Then, for any δ with $0 < \delta < r$, the path ω must contain a segment joining the spherical shell of radius δ to the spherical shell of radius r, and lying between these two shells. The length of this segment will be at least $r - \delta$; hence if we let δ go to zero, the length of ω will be at least r. If $\omega([0, 1]) \neq \gamma([0, 1])$, we easily obtain a strict inequality. $\square$

Here is an important consequence of Theorem 12.8.

Corollary 12.9 *Let (M, g) be a Riemannian manifold. If $\omega\colon [0, b] \to M$ is any curve parametrized by arc-length and ω has length less than or equal to the length of any other curve from $\omega(0)$ to $\omega(b)$, then ω is a geodesic.*

Proof. Consider any segment of ω lying within an open set, W, as above, and having length $< \epsilon$. By Theorem 12.8, this segment must be a geodesic. Hence, the entire curve is a geodesic. □

Definition 12.6 Let (M, g) be a Riemannian manifold. A geodesic, $\gamma\colon [a, b] \to M$, is *minimal* iff its length is less than or equal to the length of any other piecewise smooth curve joining its endpoints.

Theorem 12.8 asserts that any sufficiently small segment of a geodesic is minimal. On the other hand, a long geodesic may not be minimal. For example, a great circle arc on the unit sphere is a geodesic. If such an arc has length greater than π, then it is not minimal. Minimal geodesics are generally not unique. For example, any two antipodal points on a sphere are joined by an infinite number of minimal geodesics.

A *broken geodesic* is a piecewise smooth curve as in Definition 12.1, where each curve segment is a geodesic.

Proposition 12.10 *A Riemannian manifold, (M, g), is connected iff any two points of M can be joined by a broken geodesic.*

Proof. Assume M is connected, pick any $p \in M$, and let $S_p \subseteq M$ be the set of all points that can be connected to p by a broken geodesic. For any $q \in M$, choose a normal neighborhood, U, of q. If $q \in S_p$, then it is clear that $U \subseteq S_p$. On the other hand, if $q \notin S_p$, then $U \subseteq M - S_p$. Therefore, $S_p \neq \emptyset$ is open and closed, so $S_p = M$. The converse is obvious. □

In general, if M is connected, then it is not true that any two points are joined by a geodesic. However, this will be the case if M is geodesically complete, as we will see in the next section.

Next, we will see that a Riemannian metric induces a distance on the manifold whose induced topology agrees with the original metric.

12.3 Complete Riemannian Manifolds, the Hopf-Rinow Theorem and the Cut Locus

Every connected Riemannian manifold, (M, g), is a metric space in a natural way. Furthermore, M is a complete metric space iff M is geodesically complete. In this section, we explore briefly some properties of complete Riemannian manifolds.

Proposition 12.11 *Let (M, g) be a connected Riemannian manifold. For any two points, $p, q \in M$, let $d(p, q)$ be the greatest lower bound of the lengths of all piecewise smooth curves joining p to q. Then, d is a metric on M and the topology of the metric space, (M, d), coincides with the original topology of M.*

A proof of the above proposition can be found in Gallot, Hulin and Lafontaine [60] (Chapter 2, Proposition 2.91) or O'Neill [119] (Chapter 5, Proposition 18).

The distance, d, is often called the *Riemannian distance* on M. For any $p \in M$ and any $\epsilon > 0$, the *metric ball of center p and radius ϵ* is the subset, $B_\epsilon(p) \subseteq M$, given by

$$B_\epsilon(p) = \{q \in M \mid d(p, q) < \epsilon\}.$$

The next proposition follows easily from Proposition 12.4 (Milnor [106], Section 10, Corollary 10.8).

Proposition 12.12 *Let (M, g) be a connected Riemannian manifold. For any compact subset, $K \subseteq M$, there is a number $\delta > 0$ so that any two points, $p, q \in K$, with distance $d(p, q) < \delta$ are joined by a unique geodesic of length less than δ. Furthermore, this geodesic is minimal and depends smoothly on its endpoints.*

Recall from Definition 12.4 that (M, g) is geodesically complete iff the exponential map, $v \mapsto \exp_p(v)$, is defined for all $p \in M$ and for all $v \in T_pM$. We now prove the following important theorem due to Hopf and Rinow (1931):

Theorem 12.13 *(Hopf-Rinow) Let (M, g) be a connected Riemannian manifold. If there is a point, $p \in M$, such that $\exp_p$ is defined on the entire tangent space, T_pM, then any point, $q \in M$, can be joined to p by a minimal geodesic. As a consequence, if M is geodesically complete, then any two points of M can be joined by a minimal geodesic.*

Proof. We follow Milnor's proof in [106], Chapter 10, Theorem 10.9. Pick any two points, $p, q \in M$ and let $r = d(p, q)$. By Proposition 12.4, there is some open subset, W, with $p \in W$ and some $\epsilon > 0$ so that any two points of W are joined by a unique geodesic and the exponential map is a diffeomorphism between the open ball, $B(0, \epsilon)$, and its image, $U_p = \exp_p(B(0, \epsilon))$. For $\delta < \epsilon$, let $S = \exp_p(S(0, \delta))$, where $S(0, \delta)$ is the sphere of radius δ. Since $S \subseteq U_p$ is compact, there is some point,

$$p_0 = \exp_p(\delta v), \qquad \text{with } \|v\| = 1,$$

on S for which the distance to q is minimized. We will prove that

$$\exp_p(rv) = q,$$

which will imply that the geodesic, γ, given by $\gamma(t) = \exp_p(tv)$ is actually a minimal geodesic from p to q (with $t \in [0, r]$). Here, we use the fact that the exponential $\exp_p$ is defined everywhere on T_pM.

The proof amounts to showing that a point which moves along the geodesic γ must get closer and closer to q. In fact, for each $t \in [\delta, r]$, we prove

$$d(\gamma(t), q) = r - t. \tag{$*_t$}$$

We get the proof by setting $t = r$.

First, we prove $(*_\delta)$. Since every path from p to q must pass through S, by the choice of p_0, we have

$$r = d(p, q) = \min_{s \in S}\{d(p, s) + d(s, q)\} = \delta + d(p_0, q).$$

Therefore, $d(p_0, q) = r - \delta$ and since $p_0 = \gamma(\delta)$, this proves $(*_\delta)$.

Define $t_0 \in [\delta, r]$ by

$$t_0 = \sup\{t \in [\delta, r] \mid d(\gamma(t), q) = r - t\}.$$

As the set, $\{t \in [\delta, r] \mid d(\gamma(t), q) = r - t\}$, is closed, it contains its upper bound, t_0, so the equation $(*_{t_0})$ also holds. We claim that if $t_0 < r$, then we obtain a contradiction.

As we did with p, there is some small $\delta' > 0$ so that if $S' = \exp_{\gamma(t_0)}(B(0, \delta'))$, then there is some point, p_0', on S' with minimum distance from q and p_0' is joined to $\gamma(t_0)$ by a mimimal geodesic. We have

$$r - t_0 = d(\gamma(t_0), q) = \min_{s \in S'}\{d(\gamma(t_0), s) + d(s, q)\} = \delta' + d(p_0', q),$$

hence

$$d(p_0', q) = r - t_0 - \delta'. \tag{$\dagger$}$$

We claim that $p_0' = \gamma(t_0 + \delta')$.

By the triangle inequality and using ($\dagger$) (recall that $d(p, q) = r$), we have

$$d(p, p_0') \geq d(p, q) - d(p_0', q) = t_0 + \delta'.$$

But, a path of length precisely $t_0 + \delta'$ from p to p_0' is obtained by following γ from p to $\gamma(t_0)$, and then following a minimal geodesic from $\gamma(t_0)$ to p_0'. Since this broken geodesic has minimal length, by Corollary 12.9, it is a genuine (unbroken) geodesic, and so, it coincides with γ. But then, as $p_0' = \gamma(t_0 + \delta')$, equality ($\dagger$) becomes $(*_{t_0+\delta'})$, namely

$$d(\gamma(t_0 + \delta'), q) = r - (t_0 + \delta'),$$

contradicting the maximality of t_0. Therefore, we must have $t_0 = r$ and $q = \exp_p(rv)$, as desired. $\square$

Remark: Theorem 12.13 is proved is every decent book on Riemannian geometry. Among those, we mention Gallot, Hulin and Lafontaine [60], Chapter 2, Theorem 2.103 and O'Neill [119], Chapter 5, Lemma 24.

Theorem 12.13 implies the following result (often known as the *Hopf-Rinow Theorem*):

Theorem 12.14 *Let (M, g) be a connected, Riemannian manifold. The following statements are equivalent:*

(1) The manifold (M, g) is geodesically complete, that is, for every $p \in M$, every geodesic through p can be extended to a geodesic defined on all of $\mathbb{R}$.

(2) For every point, $p \in M$, the map $\exp_p$ is defined on the entire tangent space, T_pM.

(3) There is a point, $p \in M$, such that $\exp_p$ is defined on the entire tangent space, T_pM.

(4) Any closed and bounded subset of the metric space, (M, d), is compact.

(5) The metric space, (M, d), is complete (that is, every Cauchy sequence converges).

Proofs of Theorem 12.14 can be found in Gallot, Hulin and Lafontaine [60], Chapter 2, Corollary 2.105 and O'Neill [119], Chapter 5, Theorem 21.

In view of Theorem 12.14, a connected Riemannian manifold, (M, g), is geodesically complete iff the metric space, (M, d), is complete. We will refer simply to M as a *complete Riemannian manifold* (it is understood that M is connected). Also, by (4), every compact, Riemannian manifold is complete. If we remove any point, p, from a Riemannian manifold, M, then $M - \{p\}$ is not complete since every geodesic that formerly went through p yields a geodesic that can't be extended.

Assume (M, g) is a complete Riemannian manifold. Given any point, $p \in M$, it is interesting to consider the subset, $\mathcal{U}_p \subseteq T_pM$, consisting of all $v \in T_pM$ such that the geodesic

$$t \mapsto \exp_p(tv)$$

is a minimal geodesic up to $t = 1 + \epsilon$, for some $\epsilon > 0$. The subset $\mathcal{U}_p$ is open and star-shaped and it turns out that $\exp_p$ is a diffeomorphism from $\mathcal{U}_p$ onto its image, $\exp_p(\mathcal{U}_p)$, in M. The left-over part, $M - \exp_p(\mathcal{U}_p)$ (if nonempty), is actually equal to $\exp_p(\partial\mathcal{U}_p)$ and it is an important subset of M called the *cut locus of* p. The following proposition is needed to establish properties of the cut locus:

Proposition 12.15 *Let (M, g) be a complete Riemannian manifold. For any geodesic, $\gamma\colon [0, a] \to M$, from $p = \gamma(0)$ to $q = \gamma(a)$, the following properties hold:*

(i) If there is no geodesic shorter than γ between p and q, then γ is minimal on $[0, a]$.

(ii) If there is another geodesic of the same length as γ between p and q, then γ is no longer minimal on any larger interval, $[0, a + \epsilon]$.

(iii) If γ is minimal on any interval, I, then γ is also minimal on any subinterval of I.

Proof. Part (iii) is an immediate consequence of the triangle inequality. As M is complete, by the Hopf-Rinow Theorem, there is a minimal geodesic from p to q, so γ must be minimal too. This proves part (i). Part (ii) is proved in Gallot, Hulin and Lafontaine [60], Chapter 2, Corollary 2.111. $\square$

Again, assume (M, g) is a complete Riemannian manifold and let $p \in M$ be any point. For every $v \in T_pM$, let

$$I_v = \{s \in \mathbb{R} \cup \{\infty\} \mid \text{the geodesic} \quad t \mapsto \exp_p(tv) \quad \text{is minimal on } [0, s]\}.$$

It is easy to see that I_v is a closed interval, so $I_v = [0, \rho(v)]$ (with $\rho(v)$ possibly infinite). It can be shown that if $w = \lambda v$, then $\rho(v) = \lambda\rho(w)$, so we can restrict our attention to unit vectors, v. It can also be shown that the map, $\rho\colon S^{n-1} \to \mathbb{R}$, is continuous, where S^{n-1} is the unit sphere of center 0 in T_pM, and that $\rho(v)$ is bounded below by a strictly positive number.

Definition 12.7 Let (M, g) be a complete Riemannian manifold and let $p \in M$ be any point. Define $\mathcal{U}_p$ by

$$\mathcal{U}_p = \left\{ v \in T_pM \;\middle|\; \rho\left(\frac{v}{\|v\|}\right) > \|v\| \right\} = \{v \in T_pM \mid \rho(v) > 1\}$$

and the *cut locus* of p by

$$\mathrm{Cut}(p) = \exp_p(\partial\mathcal{U}_p) = \{\exp_p(\rho(v)v) \mid v \in S^{n-1}\}.$$

The set $\mathcal{U}_p$ is open and star-shaped. The boundary, $\partial\mathcal{U}_p$, of $\mathcal{U}_p$ in T_pM is sometimes called the *tangential cut locus* of p and is denoted $\widetilde{\mathrm{Cut}}(p)$.

Remark: The cut locus was first introduced for convex surfaces by Poincaré (1905) under the name *ligne de partage*. According to Do Carmo [50] (Chapter 13, Section 2), for Riemannian manifolds, the cut locus was introduced by J.H.C. Whitehead (1935). But it was Klingenberg (1959) who revived the interest in the cut locus and showed its usefuleness.

Proposition 12.16 *Let (M, g) be a complete Riemannian manifold. For any point, $p \in M$, the sets $\exp_p(\mathcal{U}_p)$ and $\mathrm{Cut}(p)$ are disjoint and*

$$M = \exp_p(\mathcal{U}_p) \cup \mathrm{Cut}(p).$$

Proof. From the Hopf-Rinow Theorem, for every $q \in M$, there is a minimal geodesic, $t \mapsto \exp_p(vt)$ such that $\exp_p(v) = q$. This shows that $\rho(v) \geq 1$, so $v \in \overline{\mathcal{U}_p}$ and

$$M = \exp_p(\mathcal{U}_p) \cup \mathrm{Cut}(p).$$

It remains to show that this is a disjoint union. Assume $q \in \exp_p(\mathcal{U}_p) \cap \mathrm{Cut}(p)$. Since $q \in \exp_p(\mathcal{U}_p)$, there is a geodesic, γ, such that $\gamma(0) = p$, $\gamma(a) = q$ and γ is minimal on $[0, a + \epsilon]$, for some $\epsilon > 0$. On the other hand, as $q \in \mathrm{Cut}(p)$, there is some geodesic, $\widetilde{\gamma}$, with $\widetilde{\gamma}(0) = p$, $\widetilde{\gamma}(b) = q$, $\widetilde{\gamma}$ minimal on $[0, b]$, but $\widetilde{\gamma}$ not minimal after b. As γ and $\widetilde{\gamma}$ are both minimal from p to q, they have the same length from p to q. But then, as γ and $\widetilde{\gamma}$ are distinct, by Proposition 12.15 (ii), the geodesic γ can't be minimal after q, a contradiction. $\square$

Observe that the injectivity radius, $i(p)$, of M at p is equal to the distance from p to the cut locus of p:

$$i(p) = d(p, \mathrm{Cut}(p)) = \inf_{q \in \mathrm{Cut}(p)} d(p, q).$$

Consequently, the injectivity radius, $i(M)$, of M is given by

$$i(M) = \inf_{p \in M} d(p, \mathrm{Cut}(p)).$$

If M is compact, it can be shown that $i(M) > 0$. It can also be shown using Jacobi fields that $\exp_p$ is a diffeomorphism from $\mathcal{U}_p$ onto its image, $\exp_p(\mathcal{U}_p)$. Thus, $\exp_p(\mathcal{U}_p)$ is diffeomorphic to an open ball in $\mathbb{R}^n$ (where $n = \dim(M)$) and the cut locus is closed. Hence, the manifold, M, is obtained by gluing together an open n-ball onto the cut locus of a point. In some sense the topology of M is "contained" in its cut locus.

Given any sphere, S^{n-1}, the cut locus of any point, p, is its antipodal point, $\{-p\}$. For more examples, consult Gallot, Hulin and Lafontaine [60] (Chapter 2, Section 2C7), Do Carmo [50] (Chapter 13, Section 2) or Berger [16] (Chapter 6). In general, the cut locus is very hard to compute. In fact, according to Berger [16], even for an ellipsoid, the determination of the cut locus of an arbitrary point is still a matter of conjecture!

12.4 The Calculus of Variations Applied to Geodesics; The First Variation Formula

Given a Riemannian manifold, (M, g), the path space, $\Omega(p, q)$, was introduced in Definition 12.1. It is an "infinite dimensional" manifold. By analogy with finite dimensional manifolds, we define a kind of tangent space to $\Omega(p, q)$ at a "point" ω.

Definition 12.8 For every "point", $\omega \in \Omega(p, q)$, we define the "*tangent space*", $T_\omega\Omega(p, q)$, of $\Omega(p, q)$ at ω, to be the space of all piecewise smooth vector fields, W, along ω, for which $W(0) = W(1) = 0$ (we may assume that our paths, ω, are parametrized over $[0, 1]$).

Now, if $F\colon \Omega(p, q) \to \mathbb{R}$ is a real-valued function on $\Omega(p, q)$, it is natural to ask what the induced "tangent map",

$$dF_\omega\colon T_\omega\Omega(p, q) \to \mathbb{R},$$

should mean (here, we are identifying $T_{F(\omega)}\mathbb{R}$ with $\mathbb{R}$). Observe that $\Omega(p,q)$ is not even a topological space so the answer is far from obvious! In the case where $f\colon M \to \mathbb{R}$ is a function on a manifold, there are various equivalent ways to define df, one of which involves curves. For every $v \in T_pM$, if $\alpha\colon (-\epsilon,\epsilon) \to M$ is a curve such that $\alpha(0) = p$ and $\alpha'(0) = v$, then we know that

$$df_p(v) = \left.\frac{d(f(\alpha(t)))}{dt}\right|_{t=0}.$$

We may think of α as a small *variation* of p. Recall that p is a *critical point* of f iff $df_p(v) = 0$, for all $v \in T_pM$.

Rather than attempting to define dF_ω (which requires some conditions on F), we will mimic what we did with functions on manifolds and define what is a *critical path* of a function, $F\colon \Omega(p,q) \to \mathbb{R}$, using the notion of *variation*. Now, geodesics from p to q are special paths in $\Omega(p,q)$ and they turn out to be the critical paths of the *energy function*,

$$E_a^b(\omega) = \int_a^b \|\omega'(t)\|^2\, dt,$$

where $\omega \in \Omega(p,q)$, and $0 \leq a < b \leq 1$.

Definition 12.9 Given any path, $\omega \in \Omega(p,q)$, a *variation of* ω *(keeping endpoints fixed)* is a function, $\widetilde{\alpha}\colon (-\epsilon,\epsilon) \to \Omega(p,q)$, for some $\epsilon > 0$, such that

(1) $\widetilde{\alpha}(0) = \omega$

(2) There is a subdivision, $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$ of $[0,1]$ so that the map

$$\alpha\colon (-\epsilon,\epsilon) \times [0,1] \to M$$

defined by $\alpha(u,t) = \widetilde{\alpha}(u)(t)$ is smooth on each strip $(-\epsilon,\epsilon)\times[t_i,t_{i+1}]$, for $i = 0,\ldots,k-1$.

If U is an open subset of $\mathbb{R}^n$ containing the origin and if we replace $(-\epsilon,\epsilon)$ by U in the above, then $\widetilde{\alpha}\colon U \to \Omega(p,q)$ is called an *n-parameter variation* of ω.

The function α is also called a *variation* of ω. Since each $\widetilde{\alpha}(u)$ belongs to $\Omega(p,q)$, note that

$$\alpha(u,0) = p, \quad \alpha(u,1) = q, \quad \text{for all } u \in (-\epsilon,\epsilon).$$

The function, $\widetilde{\alpha}$, may be considered as a "smooth path" in $\Omega(p,q)$, since for every $u \in (-\epsilon,\epsilon)$, the map $\widetilde{\alpha}(u)$ is a curve in $\Omega(p,q)$ called a *curve in the variation (or longitudinal curve of the variation)*. The "velocity vector", $\frac{d\widetilde{\alpha}}{du}(0) \in T_\omega\Omega(p,q)$, is defined to be the vector field, W, along ω, given by

$$W_t = \frac{d\widetilde{\alpha}}{du}(0)_t = \frac{\partial\alpha}{\partial u}(0,t),$$

Clearly, $W \in T_\omega\Omega(p,q)$. In particular, $W(0) = W(1) = 0$. The vector field, W, is also called the *variation vector field* associated with the variation α.

Besides the curves in the variation, $\widetilde{\alpha}(u)$ (with $u \in (-\epsilon, \epsilon)$), for every $t \in [0, 1]$, we have a curve, $\alpha_t \colon (-\epsilon, \epsilon) \to M$, called a *transversal curve of the variation*, defined by

$$\alpha_t(u) = \widetilde{\alpha}(u)(t),$$

and W_t is equal to the velocity vector, $\alpha_t'(0)$, at the point $\omega(t) = \alpha_t(0)$. For ϵ sufficiently small, the vector field, W_t, is an infinitesimal model of the variation $\widetilde{\alpha}$.

We can show that for any $W \in T_\omega\Omega(p, q)$ there is a variation, $\widetilde{\alpha} \colon (-\epsilon, \epsilon) \to \Omega(p, q)$, which satisfies the conditions

$$\widetilde{\alpha}(0) = \omega, \qquad \frac{d\widetilde{\alpha}}{du}(0) = W.$$

Here is a sketch of the proof: By the compactness of $\omega([0, 1])$, it is possible to find a $\delta > 0$ so that $\exp_{\omega(t)}$ is defined for all $t \in [0, 1]$ and all $v \in T_{\omega(t)}M$, with $\|v\| < \delta$. Then, if

$$N = \max_{t \in [0,1]} \|W_t\|,$$

for any ϵ such that $0 < \epsilon < \frac{\delta}{N}$, it can be shown that

$$\widetilde{\alpha}(u)(t) = \exp_{\omega(t)}(uW_t)$$

works (for details, see Do Carmo [50], Chapter 9, Proposition 2.2).

As we said earlier, given a function, $F \colon \Omega(p, q) \to \mathbb{R}$, we do not attempt to define the differential, dF_ω, but instead, the notion of critical path.

Definition 12.10 Given a function, $F \colon \Omega(p, q) \to \mathbb{R}$, we say that a path, $\omega \in \Omega(p, q)$, is a *critical path* for F iff

$$\left.\frac{dF(\widetilde{\alpha}(u))}{du}\right|_{u=0} = 0,$$

for every variation, $\widetilde{\alpha}$, of ω (which implies that the derivative $\left.\frac{dF(\widetilde{\alpha}(u))}{du}\right|_{u=0}$ is defined for every variation, $\widetilde{\alpha}$, of ω).

For example, if F takes on its minimum on a path ω_0 and if the derivatives $\frac{dF(\widetilde{\alpha}(u))}{du}$ are all defined, then ω_0 is a critical path of F.

We will apply the above to two functions defined on $\Omega(p, q)$:

(1) The *energy function* (also called *action integral*):

$$E_a^b(\omega) = \int_a^b \|\omega'(t)\|^2 \, dt.$$

(We write $E = E_0^1$.)

(2) The *arc-length function*,

$$L_a^b(\omega) = \int_a^b \|\omega'(t)\|\, dt.$$

The quantities $E_a^b(\omega)$ and $L_a^b(\omega)$ can be compared as follows: if we apply the Cauchy-Schwarz's inequality,

$$\left(\int_a^b f(t)g(t)dt\right)^2 \leq \left(\int_a^b f^2(t)dt\right)\left(\int_a^b g^2(t)dt\right)$$

with $f(t) \equiv 1$ and $g(t) = \|\omega'(t)\|$, we get

$$(L_a^b(\omega))^2 \leq (b-a)E_a^b,$$

where equality holds iff g is constant; that is, iff the parameter t is proportional to arc-length.

Now, suppose that there exists a minimal geodesic, γ, from p to q. Then,

$$E(\gamma) = L(\gamma)^2 \leq L(\omega)^2 \leq E(\omega),$$

where the equality $L(\gamma)^2 = L(\omega)^2$ holds only if ω is also a minimal geodesic, possibly reparametrized. On the other hand, the equality $L(\omega) = E(\omega)^2$ can hold only if the parameter is proportional to arc-length along ω. This proves that $E(\gamma) < E(\omega)$ unless ω is also a minimal geodesic. We just proved:

Proposition 12.17 *Let (M, g) be a complete Riemannian manifold. For any two points, $p, q \in M$, if $d(p, q) = \delta$, then the energy function, $E\colon \Omega(p, q) \to \mathbb{R}$, takes on its minimum, δ^2, precisely on the set of minimal geodesics from p to q.*

Next, we are going to show that the critical paths of the energy function are exactly the geodesics. For this, we need the *first variation formula*.

Let $\widetilde{\alpha}\colon (-\epsilon, \epsilon) \to \Omega(p, q)$ be a variation of ω and let

$$W_t = \frac{\partial \alpha}{\partial u}(0, t)$$

be its associated variation vector field. Furthermore, let

$$V_t = \frac{d\omega}{dt} = \omega'(t),$$

the velocity vector of ω and

$$\Delta_t V = V_{t_+} - V_{t_-},$$

the discontinuity in the velocity vector at t, which is nonzero only for $t = t_i$, with $0 < t_i < 1$ (see the definition of $\gamma'((t_i)_+)$ and $\gamma'((t_i)_-)$ just after Definition 12.1).

Theorem 12.18 *(First Variation Formula) For any path, $\omega \in \Omega(p, q)$, we have*

$$\frac{1}{2}\,\frac{dE(\widetilde{\alpha}(u))}{du}\bigg|_{u=0} = -\sum_i \langle W_t, \Delta_t V\rangle - \int_0^1 \left\langle W_t, \frac{D}{dt}V_t\right\rangle dt,$$

where $\widetilde{\alpha}\colon (-\epsilon, \epsilon) \to \Omega(p, q)$ is any variation of ω.

Proof. (After Milnor, see [106], Chapter II, Section 12, Theorem 12.2.) By Proposition 11.24, we have

$$\frac{\partial}{\partial u}\left\langle \frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial t}\right\rangle = 2\left\langle \frac{D}{\partial u}\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial t}\right\rangle.$$

Therefore,

$$\frac{dE(\widetilde{\alpha}(u))}{du} = \frac{d}{du}\int_0^1 \left\langle \frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial t}\right\rangle dt = 2\int_0^1 \left\langle \frac{D}{\partial u}\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial t}\right\rangle dt.$$

Now, because we are using the Levi-Civita connection, which is torsion-free, it is not hard to prove that

$$\frac{D}{\partial t}\frac{\partial\alpha}{\partial u} = \frac{D}{\partial u}\frac{\partial\alpha}{\partial t},$$

so

$$\frac{dE(\widetilde{\alpha}(u))}{du} = 2\int_0^1 \left\langle \frac{D}{\partial t}\frac{\partial\alpha}{\partial u}, \frac{\partial\alpha}{\partial t}\right\rangle dt.$$

We can choose $0 = t_0 < t_1 < \cdots < t_k = 1$ so that α is smooth on each strip $(-\epsilon, \epsilon) \times [t_{i-1}, t_i]$. Then, we can "integrate by parts" on $[t_{i-1}, t_i]$ as follows: The equation

$$\frac{\partial}{\partial t}\left\langle \frac{\partial\alpha}{\partial u}, \frac{\partial\alpha}{\partial t}\right\rangle = \left\langle \frac{D}{\partial t}\frac{\partial\alpha}{\partial u}, \frac{\partial\alpha}{\partial t}\right\rangle + \left\langle \frac{\partial\alpha}{\partial u}, \frac{D}{\partial t}\frac{\partial\alpha}{\partial t}\right\rangle$$

implies that

$$\int_{t_{i-1}}^{t_i} \left\langle \frac{D}{\partial t}\frac{\partial\alpha}{\partial u}, \frac{\partial\alpha}{\partial t}\right\rangle dt = \left\langle \frac{\partial\alpha}{\partial u}, \frac{\partial\alpha}{\partial t}\right\rangle\bigg|_{t=(t_{i-1})_+}^{t=(t_i)_-} - \int_{t_{i-1}}^{t_i} \left\langle \frac{\partial\alpha}{\partial u}, \frac{D}{\partial t}\frac{\partial\alpha}{\partial t}\right\rangle dt.$$

Adding up these formulae for $i = 1, \ldots k-1$ and using the fact that $\frac{\partial\alpha}{\partial u} = 0$ for $t = 0$ and $t = 1$, we get

$$\frac{1}{2}\,\frac{dE(\widetilde{\alpha}(u))}{du} = -\sum_{i=1}^{k-1}\left\langle \frac{\partial\alpha}{\partial u}, \Delta_{t_i}\frac{\partial\alpha}{\partial t}\right\rangle - \int_0^1 \left\langle \frac{\partial\alpha}{\partial u}, \frac{D}{\partial t}\frac{\partial\alpha}{\partial t}\right\rangle dt.$$

Setting $u = 0$, we obtain the formula

$$\frac{1}{2}\,\frac{dE(\widetilde{\alpha}(u))}{du}\bigg|_{u=0} = -\sum_i \langle W_t, \Delta_t V\rangle - \int_0^1 \left\langle W_t, \frac{D}{dt}V_t\right\rangle dt,$$

as claimed. $\square$

Intuitively, the first term on the right-hand side shows that varying the path ω in the direction of decreasing "kink" tends to decrease E. The second term shows that varying the curve in the direction of its acceleration vector, $\frac{D}{dt}\omega'(t)$, also tends to reduce E.

A geodesic, γ, (parametrized over $[0,1]$) is smooth on the entire interval $[0,1]$ and its acceleration vector, $\frac{D}{dt}\gamma'(t)$, is identically zero along γ. This gives us half of

Theorem 12.19 *Let (M,g) be a Riemanian manifold. For any two points, $p,q \in M$, a path, $\omega \in \Omega(p,q)$ (parametrized over $[0,1]$), is critical for the energy function, E, iff ω is a geodesic.*

Proof. From the first variation formula, it is clear that a geodesic is a critical path of E.

Conversely, assume ω is a critical path of E. There is a variation, $\widetilde{\alpha}$, of ω such that its associated variation vector field is of the form

$$W(t) = f(t)\frac{D}{dt}\omega'(t),$$

with $f(t)$ smooth and positive except that it vanishes at the t_i's. For this variation, we get

$$\frac{1}{2}\left.\frac{dE(\widetilde{\alpha}(u))}{du}\right|_{u=0} = -\int_0^1 f(t)\left\langle \frac{D}{dt}\omega'(t), \frac{D}{dt}\gamma'(t)\right\rangle dt.$$

This expression is zero iff

$$\frac{D}{dt}\omega'(t) = 0 \qquad \text{on } [0,1].$$

Hence, the restriction of ω to each $[t_i, t_{i+1}]$ is a geodesic.

It remains to prove that ω is smooth on the entire interval $[0,1]$. For this, pick a variation $\widetilde{\alpha}$ such that

$$W(t_i) = \Delta_{t_i}V.$$

Then, we have

$$\frac{1}{2}\left.\frac{dE(\widetilde{\alpha}(u))}{du}\right|_{u=0} = -\sum_{i=1}^{k}\langle \Delta_{t_i}V, \Delta_{t_i}V\rangle.$$

If the above expression is zero, then $\Delta_{t_i}V = 0$ for $i = 1, \ldots, k-1$, which means that ω is C^1 everywhere on $[0,1]$. By the uniqueness theorem for ODE's, ω must be smooth everywhere on $[0,1]$, and thus, it is an unbroken geodesic. $\square$

Remark: If $\omega \in \Omega(p,q)$ is parametrized by arc-length, it is easy to prove that

$$\left.\frac{dL(\widetilde{\alpha}(u))}{du}\right|_{u=0} = \frac{1}{2}\left.\frac{dE(\widetilde{\alpha}(u))}{du}\right|_{u=0}.$$

As a consequence, a path, $\omega \in \Omega(p, q)$ is critical for the arc-length function, L, iff it can be reparametrized so that it is a geodesic (see Gallot, Hulin and Lafontaine [60], Chapter 3, Theorem 3.31).

In order to go deeper into the study of geodesics we need Jacobi fields and the "second variation formula", both involving a curvature term. Therefore, we now proceed with a more thorough study of curvature on Riemannian manifolds.