Chapter 22

Tensor Algebras, Symmetric Algebras and Exterior Algebras

22.1 Tensors Products

We begin by defining tensor products of vector spaces over a field and then we investigate some basic properties of these tensors, in particular the existence of bases and duality. After this, we investigate special kinds of tensors, namely, symmetric tensors and skew-symmetric tensors. Tensor products of modules over a commutative ring with identity will be discussed very briefly. They show up naturally when we consider the space of sections of a tensor product of vector bundles.

Given a linear map, $f\colon E \to F$, we know that if we have a basis, $(u_i)_{i\in I}$, for E, then f is completely determined by its values, $f(u_i)$, on the basis vectors. For a multilinear map, $f\colon E^n \to F$, we don't know if there is such a nice property but it would certainly be very useful.

In many respects, tensor products allow us to define multilinear maps in terms of their action on a suitable basis. The crucial idea is to *linearize*, that is, to create a new vector space, $E^{\otimes n}$, such that the multilinear map, $f\colon E^n \to F$, is turned into a *linear map*, $f_{\otimes}\colon E^{\otimes n} \to F$, which is equivalent to f in a strong sense. If in addition, f is symmetric, then we can define a symmetric tensor power, $\mathrm{Sym}^n(E)$, and every symmetric multilinear map, $f\colon E^n \to F$, is turned into a *linear map*, $f_{\odot}\colon \mathrm{Sym}^n(E) \to F$, which is equivalent to f in a strong sense. Similarly, if f is alternating, then we can define a skew-symmetric tensor power, $\bigwedge^n(E)$, and every alternating multilinear map is turned into a *linear map*, $f_{\wedge}\colon \bigwedge^n(E) \to F$, which is equivalent to f in a strong sense.

Tensor products can be defined in various ways, some more abstract than others. We tried to stay down to earth, without excess!

Let K be a given field, and let $E_1, \ldots, E_n$ be $n \geq 2$ given vector spaces. For any vector space, F, recall that a map, $f\colon E_1 \times \cdots \times E_n \to F$, is *multilinear* iff it is linear in each of its argument, that is,

$$\begin{aligned} f(u_1, \ldots u_{i_1}, v + w, u_{i+1}, \ldots, u_n) &= f(u_1, \ldots u_{i_1}, v, u_{i+1}, \ldots, u_n) \\ &\quad + f(u_1, \ldots u_{i_1}, w, u_{i+1}, \ldots, u_n) \\ f(u_1, \ldots u_{i_1}, \lambda v, u_{i+1}, \ldots, u_n) &= \lambda f(u_1, \ldots u_{i_1}, v, u_{i+1}, \ldots, u_n), \end{aligned}$$

for all $u_j \in E_j$ $(j \neq i)$, all $v, w \in E_i$ and all $\lambda \in K$, for $i = 1 \ldots, n$.

The set of multilinear maps as above forms a vector space denoted $\mathrm{L}(E_1, \ldots, E_n; F)$ or $\mathrm{Hom}(E_1, \ldots, E_n; F)$. When $n = 1$, we have the vector space of linear maps, $\mathrm{L}(E, F)$ or $\mathrm{Hom}(E, F)$. (To be very precise, we write $\mathrm{Hom}_K(E_1, \ldots, E_n; F)$ and $\mathrm{Hom}_K(E, F)$.) As usual, the *dual space, E^*, of E* is defined by $E^* = \mathrm{Hom}(E, K)$.

Before proceeding any further, we recall a basic fact about pairings. We will use this fact to deal with dual spaces of tensors.

Definition 22.1 Given two vector spaces, E and F, a map, $(-, -)\colon E \times F \to K$, is a *nondegenerate pairing* iff it is bilinear and iff $(u, v) = 0$ for all $v \in F$ implies $u = 0$ and $(u, v) = 0$ for all $u \in E$ implies $v = 0$. A nondegenerate pairing induces two linear maps, $\varphi\colon E \to F^*$ and $\psi\colon F \to E^*$, defined by

$$\begin{aligned} \varphi(u)(y) &= (u, y) \\ \psi(v)(x) &= (x, v), \end{aligned}$$

for all $u, x \in E$ and all $v, y \in F$.

Proposition 22.1 *For every nondegenerate pairing, $(-, -)\colon E \times F \to K$, the induced maps $\varphi\colon E \to F^*$ and $\psi\colon F \to E^*$ are linear and injective. Furthermore, if E and F are finite dimensional, then $\varphi\colon E \to F^*$ and $\psi\colon F \to E^*$ are bijective.*

Proof. The maps $\varphi\colon E \to F^*$ and $\psi\colon F \to E^*$ are linear because $u, v \mapsto (u, v)$ is bilinear. Assume that $\varphi(u) = 0$. This means that $\varphi(u)(y) = (u, y) = 0$ for all $y \in F$ and as our pairing is nondegenerate, we must have $u = 0$. Similarly, ψ is injective. If E and F are finite dimensional, then $\dim(E) = \dim(E^*)$ and $\dim(F) = \dim(F^*)$. However, the injectivity of φ and ψ implies that that $\dim(E) \leq \dim(F^*)$ and $\dim(F) \leq \dim(E^*)$. Consequently $\dim(E) \leq \dim(F)$ and $\dim(F) \leq \dim(E)$, so $\dim(E) = \dim(F)$. Therefore, $\dim(E) = \dim(F^*)$ and φ is bijective (and similarly $\dim(F) = \dim(E^*)$ and ψ is bijective). $\square$

Proposition 22.1 shows that when E and F are finite dimensional, a nondegenerate pairing induces *canonical isomorphims* $\varphi\colon E \to F^*$ and $\psi\colon F \to E^*$, that is, isomorphisms that do not depend on the choice of bases. An important special case is the case where $E = F$ and we have an inner product (a symmetric, positive definite bilinear form) on E.

Remark: When we use the term "canonical isomorphism" we mean that such an isomorphism is defined independently of any choice of bases. For example, if E is a finite dimensional vector space and $(e_1, \ldots, e_n)$ is any basis of E, we have the dual basis, $(e_1^*, \ldots, e_n^*)$, of E^* (where, $e_i^*(e_j) = \delta_{i\,j}$) and thus, the map $e_i \mapsto e_i^*$ is an isomorphism between E and E^*. This isomorphism is *not* canonical.

On the other hand, if $\langle -, - \rangle$ is an inner product on E, then Proposition 22.1 shows that the nondegenerate pairing, $\langle -, - \rangle$, induces a canonical isomorphism between E and E^*. This isomorphism is often denoted $\flat \colon E \to E^*$ and we usually write $u^\flat$ for $\flat(u)$, with $u \in E$. Given any basis, $(e_1, \ldots, e_n)$, of E (not necessarily orthonormal), if we let $g_{ij} = (e_i, e_j)$, then for every $u = \sum_{i=1}^n u_i e_i$, since $u^\flat(v) = \langle u, v \rangle$, for all $v \in V$, we get

$$u^\flat = \sum_{i=1}^n \omega_i e_i^*, \quad \text{with} \quad \omega_i = \sum_{j=1}^n g_{ij} u_j.$$

If we use the convention that coordinates of vectors are written using superscripts $(u = \sum_{i=1}^n u^i e_i)$ and coordinates of one-forms (covectors) are written using subscripts $(\omega = \sum_{i=1}^n \omega_i e_i^*)$, then the map, $\flat$, has the effect of lowering (flattening!) indices. The inverse of $\flat$ is denoted $\sharp \colon E^* \to E$. If we write $\omega \in E^*$ as $\omega = \sum_{i=1}^n \omega_i e_i^*$ and $\omega^\sharp \in E$ as $\omega^\sharp = \sum_{j=1}^n (\omega^\sharp)^j e_j$, since

$$\omega_i = \omega(e_i) = \langle \omega^\sharp, e_i \rangle = \sum_{j=1}^n (\omega^\sharp)^j g_{ij}, \qquad 1 \leq i \leq n,$$

we get

$$(\omega^\sharp)^i = \sum_{j=1}^n g^{ij} \omega_j,$$

where (g^{ij}) is the inverse of the matrix (g_{ij}). The inner product, $(-, -)$, on E induces an inner product on E^* also denoted $(-, -)$ and given by

$$(\omega_1, \omega_2) = (\omega_1^\sharp, \omega_2^\sharp),$$

for all $\omega_1, \omega_2 \in E^*$. Then, it is obvious that

$$(u, v) = (u^\flat, v^\flat), \quad \text{for all} \quad u, v \in E.$$

If $(e_1, \ldots, e_n)$ is a basis of E and $g_{ij} = (e_i, e_j)$, as

$$(e_i^*)^\sharp = \sum_{k=1}^n g^{ik} e_k,$$

an easy computation shows that

$$(e_i^*, e_j^*) = ((e_i^*)^\sharp, (e_j^*)^\sharp) = g^{ij},$$

that is, in the basis $(e_1^*, \ldots, e_n^*)$, the inner product on E^* is represented by the matrix (g^{ij}), the inverse of the matrix (g_{ij}).

The inner product on a finite vector space also yields a natural isomorphism between the space, $\mathrm{Hom}(E, E; K)$, of bilinear forms on E and the space, $\mathrm{Hom}(E, E)$, of linear maps from E to itself. Using this isomorphism, we can define the trace of a bilinear form in an intrinsic manner. This technique is used in differential geometry, for example, to define the divergence of a differential one-form.

Proposition 22.2 *If $\langle -, - \rangle$ is an inner product on a finite vector space, E, (over a field, K), then for every bilinear form, $f\colon E \times E \to K$, there is a unique linear map, $f^\sharp\colon E \to E$, such that*

$$f(u, v) = \langle f^\sharp(u), v \rangle, \quad \text{for all } u, v \in E.$$

The map, $f \mapsto f^\sharp$, is a linear isomorphism between $\mathrm{Hom}(E, E; K)$ *and* $\mathrm{Hom}(E, E)$.

Proof. For every $g \in \mathrm{Hom}(E, E)$, the map given by

$$f(u, v) = \langle g(u), v \rangle, \quad u, v \in E,$$

is clearly bilinear. It is also clear that the above defines a linear map from $\mathrm{Hom}(E, E)$ to $\mathrm{Hom}(E, E; K)$. This map is injective because if $f(u, v) = 0$ for all $u, v \in E$, as $\langle -, - \rangle$ is an inner product, we get $g(u) = 0$ for all $u \in E$. Furthermore, both spaces $\mathrm{Hom}(E, E)$ and $\mathrm{Hom}(E, E; K)$ have the same dimension, so our linear map is an isomorphism. $\square$

If $(e_1, \ldots, e_n)$ is an orthonormal basis of E, then we check immediately that the trace of a linear map, g, (which is independent of the choice of a basis) is given by

$$\mathrm{tr}(g) = \sum_{i=1}^{n} \langle g(e_i), e_i \rangle,$$

where $n = \dim(E)$. We define the *trace of the bilinear form*, f, by

$$\mathrm{tr}(f) = \mathrm{tr}(f^\sharp).$$

From Proposition 22.2, $\mathrm{tr}(f)$ is given by

$$\mathrm{tr}(f) = \sum_{i=1}^{n} f(e_i, e_i),$$

for any orthonormal basis, $(e_1, \ldots, e_n)$, of E. We can also check directly that the above expression is independent of the choice of an orthonormal basis.

We will also need the following Proposition to show that various families are linearly independent.

Proposition 22.3 *Let E and F be two nontrivial vector spaces and let $(u_i)_{i\in I}$ be any family of vectors $u_i \in E$. The family, $(u_i)_{i\in I}$, is linearly independent iff for every family, $(v_i)_{i\in I}$, of vectors $v_i \in F$, there is some linear map, $f\colon E \to F$, so that $f(u_i) = v_i$, for all $i \in I$.*

Proof. Left as an exercise. □

First, we define tensor products, and then we prove their existence and uniqueness up to isomorphism.

Definition 22.2 A *tensor product* of $n \geq 2$ vector spaces $E_1, \ldots, E_n$, is a vector space T, together with a multilinear map $\varphi\colon E_1 \times \cdots \times E_n \to T$, such that, for every vector space F and for every multilinear map $f\colon E_1 \times \cdots \times E_n \to F$, there is a unique linear map $f_\otimes\colon T \to F$, with

$$f(u_1, \ldots, u_n) = f_\otimes(\varphi(u_1, \ldots, u_n)),$$

for all $u_1 \in E_1, \ldots, u_n \in E_n$, or for short

$$f = f_\otimes \circ \varphi.$$

Equivalently, there is a unique linear map $f_\otimes$ such that the following diagram commutes:

$$\begin{tikzcd}
E_1 \times \cdots \times E_n \arrow[r, "\varphi"] \arrow[dr, "f"'] & T \arrow[d, "f_\otimes"] \\
& F
\end{tikzcd}$$

First, we show that any two tensor products (T_1, φ_1) and (T_2, φ_2) for $E_1, \ldots, E_n$, are isomorphic.

Proposition 22.4 *Given any two tensor products (T_1, φ_1) and (T_2, φ_2) for $E_1, \ldots, E_n$, there is an isomorphism $h\colon T_1 \to T_2$ such that*

$$\varphi_2 = h \circ \varphi_1.$$

Proof. Focusing on (T_1, φ_1), we have a multilinear map $\varphi_2\colon E_1 \times \cdots \times E_n \to T_2$, and thus, there is a unique linear map $(\varphi_2)_\otimes\colon T_1 \to T_2$, with

$$\varphi_2 = (\varphi_2)_\otimes \circ \varphi_1.$$

Similarly, focusing now on on (T_2, φ_2), we have a multilinear map $\varphi_1\colon E_1 \times \cdots \times E_n \to T_1$, and thus, there is a unique linear map $(\varphi_1)_\otimes\colon T_2 \to T_1$, with

$$\varphi_1 = (\varphi_1)_\otimes \circ \varphi_2.$$

But then, we get

$$\varphi_1 = (\varphi_1)_\otimes \circ (\varphi_2)_\otimes \circ \varphi_1,$$

and

$$\varphi_2 = (\varphi_2)_\otimes \circ (\varphi_1)_\otimes \circ \varphi_2.$$

On the other hand, focusing on (T_1, φ_1), we have a multilinear map $\varphi_1 \colon E_1 \times \cdots \times E_n \to T_1$, but the unique linear map $h \colon T_1 \to T_1$, with

$$\varphi_1 = h \circ \varphi_1$$

is $h = \mathrm{id}$, and since $(\varphi_1)_\otimes \circ (\varphi_2)_\otimes$ is linear, as a composition of linear maps, we must have

$$(\varphi_1)_\otimes \circ (\varphi_2)_\otimes = \mathrm{id}.$$

Similarly, we must have

$$(\varphi_2)_\otimes \circ (\varphi_1)_\otimes = \mathrm{id}.$$

This shows that $(\varphi_1)_\otimes$ and $(\varphi_2)_\otimes$ are inverse linear maps, and thus, $(\varphi_2)_\otimes \colon T_1 \to T_2$ is an isomorphism between T_1 and T_2. □

Now that we have shown that tensor products are unique up to isomorphism, we give a construction that produces one.

Theorem 22.5 *Given $n \geq 2$ vector spaces $E_1, \ldots, E_n$, a tensor product $(E_1 \otimes \cdots \otimes E_n, \varphi)$ for $E_1, \ldots, E_n$ can be constructed. Furthermore, denoting $\varphi(u_1, \ldots, u_n)$ as $u_1 \otimes \cdots \otimes u_n$, the tensor product $E_1 \otimes \cdots \otimes E_n$ is generated by the vectors $u_1 \otimes \cdots \otimes u_n$, where $u_1 \in E_1, \ldots, u_n \in E_n$, and for every multilinear map $f \colon E_1 \times \cdots \times E_n \to F$, the unique linear map $f_\otimes \colon E_1 \otimes \cdots \otimes E_n \to F$ such that $f = f_\otimes \circ \varphi$, is defined by*

$$f_\otimes(u_1 \otimes \cdots \otimes u_n) = f(u_1, \ldots, u_n),$$

on the generators $u_1 \otimes \cdots \otimes u_n$ of $E_1 \otimes \cdots \otimes E_n$.

Proof. Given any set, I, viewed as an index set, let $K^{(I)}$ be the set of all functions, $f \colon I \to K$, such that $f(i) \neq 0$ only for finitely many $i \in I$. As usual, denote such a function by $(f_i)_{i \in I}$, it is a family of finite support. We make $K^{(I)}$ into a vector space by defining addition and scalar multiplication by

$$\begin{aligned} (f_i) + (g_i) &= (f_i + g_i) \\ \lambda(f_i) &= (\lambda f_i). \end{aligned}$$

The family, $(e_i)_{i \in I}$, is defined such that $(e_i)_j = 0$ if $j \neq i$ and $(e_i)_i = 1$. It is a basis of the vector space $K^{(I)}$, so that every $w \in K^{(I)}$ can be uniquely written as a finite linear combination of the e_i. There is also an injection, $\iota \colon I \to K^{(I)}$, such that $\iota(i) = e_i$ for every $i \in I$. Furthermore, it is easy to show that for any vector space, F, and for any function, $f \colon I \to F$, there is a unique linear map, $\overline{f} \colon K^{(I)} \to F$, such that

$$f = \overline{f} \circ \iota,$$

as in the following diagram:

$$\begin{tikzcd} I \arrow[r, "\iota"] \arrow[dr, "f"'] & K^{(I)} \arrow[d, "\overline{f}"] \\ & F \end{tikzcd}$$

This shows that $K^{(I)}$ is the *free vector space generated by* I. Now, apply this construction to the cartesian product, $I = E_1 \times \cdots \times E_n$, obtaining the free vector space $M = K^{(I)}$ on $I = E_1 \times \cdots \times E_n$. Since every, e_i, is uniquely associated with some n-tuple $i = (u_1, \ldots, u_n) \in E_1 \times \cdots \times E_n$, we will denote e_i by $(u_1, \ldots, u_n)$.

Next, let N be the subspace of M generated by the vectors of the following type:

$$(u_1, \ldots, u_i + v_i, \ldots, u_n) - (u_1, \ldots, u_i, \ldots, u_n) - (u_1, \ldots, v_i, \ldots, u_n),$$
$$(u_1, \ldots, \lambda u_i, \ldots, u_n) - \lambda(u_1, \ldots, u_i, \ldots, u_n).$$

We let $E_1 \otimes \cdots \otimes E_n$ be the quotient M/N of the free vector space M by N, $\pi\colon M \to M/N$ be the quotient map and set

$$\varphi = \pi \circ \iota.$$

By construction, φ is multilinear, and since π is surjective and the $\iota(i) = e_i$ generate M, since i is of the form $i = (u_1, \ldots, u_n) \in E_1 \times \cdots \times E_n$, the $\varphi(u_1, \ldots, u_n)$ generate M/N. Thus, if we denote $\varphi(u_1, \ldots, u_n)$ as $u_1 \otimes \cdots \otimes u_n$, the tensor product $E_1 \otimes \cdots \otimes E_n$ is generated by the vectors $u_1 \otimes \cdots \otimes u_n$, where $u_1 \in E_1, \ldots, u_n \in E_n$.

For every multilinear map $f\colon E_1 \times \cdots \times E_n \to F$, if a linear map $f_\otimes\colon E_1 \otimes \cdots \otimes E_n \to F$ exists such that $f = f_\otimes \circ \varphi$, since the vectors $u_1 \otimes \cdots \otimes u_n$ generate $E_1 \otimes \cdots \otimes E_n$, the map $f_\otimes$ is uniquely defined by

$$f_\otimes(u_1 \otimes \cdots \otimes u_n) = f(u_1, \ldots, u_n).$$

On the other hand, because $M = K^{(E_1 \times \cdots \times E_n)}$ is free on $I = E_1 \times \cdots \times E_n$, there is a unique linear map $\overline{f}\colon K^{(E_1 \times \cdots \times E_n)} \to F$, such that

$$f = \overline{f} \circ \iota,$$

as in the diagram below:

$$\begin{tikzcd} E_1 \times \cdots \times E_n \arrow[r, "\iota"] \arrow[dr, "f"'] & K^{(E_1 \times \cdots \times E_n)} \arrow[d, "\overline{f}"] \\ & F \end{tikzcd}$$

Because f is multilinear, note that we must have $\overline{f}(w) = 0$, for every $w \in N$. But then, $\overline{f}\colon M \to F$ induces a linear map $h\colon M/N \to F$, such that

$$f = h \circ \pi \circ \iota,$$

by defining $h([z]) = \overline{f}(z)$, for every $z \in M$, where $[z]$ denotes the equivalence class in M/N of $z \in M$:

$$\begin{tikzcd}
E_1 \times \cdots \times E_n \arrow[r, "\pi \circ \iota"] \arrow[dr, "f"'] & K^{(E_1 \times \cdots \times E_n)}/N \arrow[d, "h"] \\
& F
\end{tikzcd}$$

Indeed, the fact that $\overline{f}$ vanishes on N insures that h is well defined on M/N, and it is clearly linear by definition. However, we showed that such a linear map h is unique, and thus it agrees with the linear map $f_\otimes$ defined by

$$f_\otimes(u_1 \otimes \cdots \otimes u_n) = f(u_1, \ldots, u_n)$$

on the generators of $E_1 \otimes \cdots \otimes E_n$. $\square$

What is important about Theorem 22.5 is not so much the construction itself but the fact that it produces a tensor product with the universal mapping property with respect to multilinear maps. Indeed, Theorem 22.5 yields a canonical isomorphism,

$$\mathrm{L}(E_1 \otimes \cdots \otimes E_n, F) \cong \mathrm{L}(E_1, \ldots, E_n; F),$$

between the vector space of linear maps, $\mathrm{L}(E_1 \otimes \cdots \otimes E_n, F)$, and the vector space of multilinear maps, $\mathrm{L}(E_1, \ldots, E_n; F)$, via the linear map $- \circ \varphi$ defined by

$$h \mapsto h \circ \varphi,$$

where $h \in \mathrm{L}(E_1 \otimes \cdots \otimes E_n, F)$. Indeed, $h \circ \varphi$ is clearly multilinear, and since by Theorem 22.5, for every multilinear map, $f \in \mathrm{L}(E_1, \ldots, E_n; F)$, there is a unique linear map $f_\otimes \in \mathrm{L}(E_1 \otimes \cdots \otimes E_n, F)$ such that $f = f_\otimes \circ \varphi$, the map $- \circ \varphi$ is bijective. As a matter of fact, its inverse is the map

$$f \mapsto f_\otimes.$$

Using the "Hom" notation, the above canonical isomorphism is written

$$\mathrm{Hom}(E_1 \otimes \cdots \otimes E_n, F) \cong \mathrm{Hom}(E_1, \ldots, E_n; F).$$

Remarks:

(1) To be very precise, since the tensor product depends on the field, K, we should subscript the symbol $\otimes$ with K and write

$$E_1 \otimes_K \cdots \otimes_K E_n.$$

However, we often omit the subscript K unless confusion may arise.

(2) For $F = K$, the base field, we obtain a canonical isomorphism between the vector space $\mathrm{L}(E_1 \otimes \cdots \otimes E_n, K)$, and the vector space of multilinear forms $\mathrm{L}(E_1, \ldots, E_n; K)$. However, $\mathrm{L}(E_1 \otimes \cdots \otimes E_n, K)$ is the dual space, $(E_1 \otimes \cdots \otimes E_n)^*$, and thus, the vector space of multilinear forms $\mathrm{L}(E_1, \ldots, E_n; K)$ is canonically isomorphic to $(E_1 \otimes \cdots \otimes E_n)^*$. We write
$$\mathrm{L}(E_1, \ldots, E_n; K) \cong (E_1 \otimes \cdots \otimes E_n)^*.$$

The fact that the map $\varphi\colon E_1 \times \cdots \times E_n \to E_1 \otimes \cdots \otimes E_n$ is multilinear, can also be expressed as follows:

$$\begin{aligned} u_1 \otimes \cdots \otimes (u_i + v_i) \otimes \cdots \otimes u_n &= (u_1 \otimes \cdots \otimes u_i \otimes \cdots \otimes u_n) \\ &\quad + (u_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes u_n), \\ u_1 \otimes \cdots \otimes (\lambda u_i) \otimes \cdots \otimes u_n &= \lambda(u_1 \otimes \cdots \otimes u_i \otimes \cdots \otimes u_n). \end{aligned}$$

Of course, this is just what we wanted! Tensors in $E_1 \otimes \cdots \otimes E_n$ are also called *n-tensors*, and tensors of the form $u_1 \otimes \cdots \otimes u_n$, where $u_i \in E_i$, are called *simple (or indecomposable) n-tensors*. Those n-tensors that are not simple are often called *compound n-tensors*.

Not only do tensor products act on spaces, but they also act on linear maps (they are functors). Given two linear maps $f\colon E \to E'$ and $g\colon F \to F'$, we can define $h\colon E \times F \to E' \otimes F'$ by
$$h(u, v) = f(u) \otimes g(v).$$
It is immediately verified that h is bilinear, and thus, it induces a unique linear map
$$f \otimes g\colon E \otimes F \to E' \otimes F',$$
such that
$$(f \otimes g)(u \otimes v) = f(u) \otimes g(u).$$

If we also have linear maps $f'\colon E' \to E''$ and $g'\colon F' \to F''$, we can easily verify that the linear maps $(f' \circ f) \otimes (g' \circ g)$ and $(f' \otimes g') \circ (f \otimes g)$ agree on all vectors of the form $u \otimes v \in E \otimes F$. Since these vectors generate $E \otimes F$, we conclude that
$$(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g).$$

The generalization to the tensor product $f_1 \otimes \cdots \otimes f_n$ of $n \geq 3$ linear maps $f_i\colon E_i \to F_i$ is immediate, and left to the reader.

22.2 Bases of Tensor Products

We showed that $E_1 \otimes \cdots \otimes E_n$ is generated by the vectors of the form $u_1 \otimes \cdots \otimes u_n$. However, there vectors are not linearly independent. This situation can be fixed when considering bases, which is the object of the next proposition.

Proposition 22.6 *Given $n \geq 2$ vector spaces $E_1, \ldots, E_n$, if $(u_i^k)_{i \in I_k}$ is a basis for E_k, $1 \leq k \leq n$, then the family of vectors*

$$(u_{i_1}^1 \otimes \cdots \otimes u_{i_n}^n)_{(i_1, \ldots, i_n) \in I_1 \times \ldots \times I_n}$$

is a basis of the tensor product $E_1 \otimes \cdots \otimes E_n$.

Proof. For each k, $1 \leq k \leq n$, every $v^k \in E_k$ can be written uniquely as

$$v^k = \sum_{j \in I_k} v_j^k u_j^k,$$

for some family of scalars $(v_j^k)_{j \in I_k}$. Let F be any nontrivial vector space. We show that for every family

$$(w_{i_1, \ldots, i_n})_{(i_1, \ldots, i_n) \in I_1 \times \ldots \times I_n},$$

of vectors in F, there is some linear map $h \colon E_1 \otimes \cdots \otimes E_n \to F$, such that

$$h(u_{i_1}^1 \otimes \cdots \otimes u_{i_n}^n) = w_{i_1, \ldots, i_n}.$$

Then, by Proposition 22.3, it follows that

$$(u_{i_1}^1 \otimes \cdots \otimes u_{i_n}^n)_{(i_1, \ldots, i_n) \in I_1 \times \ldots \times I_n}$$

is linearly independent. However, since $(u_i^k)_{i \in I_k}$ is a basis for E_k, the $u_{i_1}^1 \otimes \cdots \otimes u_{i_n}^n$ also generate $E_1 \otimes \cdots \otimes E_n$, and thus, they form a basis of $E_1 \otimes \cdots \otimes E_n$.

We define the function $f \colon E_1 \times \cdots \times E_n \to F$ as follows:

$$f(\sum_{j_1 \in I_1} v_{j_1}^1 u_{j_1}^1, \ldots, \sum_{j_n \in I_n} v_{j_n}^n u_{j_n}^n) = \sum_{j_1 \in I_1, \ldots, j_n \in I_n} v_{j_1}^1 \cdots v_{j_n}^n \, w_{j_1, \ldots, j_n}.$$

It is immediately verified that f is multilinear. By the universal mapping property of the tensor product, the linear map $f_{\otimes} \colon E_1 \otimes \cdots \otimes E_n \to F$ such that $f = f_{\otimes} \circ \varphi$, is the desired map h. $\square$

In particular, when each I_k is finite and of size $m_k = \dim(E_k)$, we see that the dimension of the tensor product $E_1 \otimes \cdots \otimes E_n$ is $m_1 \cdots m_n$. As a corollary of Proposition 22.6, if $(u_i^k)_{i \in I_k}$ is a basis for E_k, $1 \leq k \leq n$, then every tensor $z \in E_1 \otimes \cdots \otimes E_n$ can be written in a unique way as

$$z = \sum_{(i_1, \ldots, i_n) \in I_1 \times \ldots \times I_n} \lambda_{i_1, \ldots, i_n} \, u_{i_1}^1 \otimes \cdots \otimes u_{i_n}^n,$$

for some unique family of scalars $\lambda_{i_1, \ldots, i_n} \in K$, all zero except for a finite number.

22.3 Some Useful Isomorphisms for Tensor Products

Proposition 22.7 *Given 3 vector spaces E, F, G, there exists unique canonical isomorphisms*

(1) $E \otimes F \simeq F \otimes E$

(2) $(E \otimes F) \otimes G \simeq E \otimes (F \otimes G) \simeq E \otimes F \otimes G$

(3) $(E \oplus F) \otimes G \simeq (E \otimes G) \oplus (F \otimes G)$

(4) $K \otimes E \simeq E$

such that respectively

(a) $u \otimes v \mapsto v \otimes u$

(b) $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w) \mapsto u \otimes v \otimes w$

(c) $(u,\, v) \otimes w \mapsto (u \otimes w,\, v \otimes w)$

(d) $\lambda \otimes u \mapsto \lambda u$.

Proof. These isomorphisms are proved using the universal mapping property of tensor products. We illustrate the proof method on (2). Fix some $w \in G$. The map

$$(u,\, v) \mapsto u \otimes v \otimes w$$

from $E \times F$ to $E \otimes F \otimes G$ is bilinear, and thus, there is a linear map $f_w\colon E \otimes F \to E \otimes F \otimes G$, such that $f_w(u \otimes v) = u \otimes v \otimes w$.

Next, consider the map

$$(z,\, w) \mapsto f_w(z),$$

from $(E \otimes F) \times G$ into $E \otimes F \otimes G$. It is easily seen to be bilinear, and thus, it induces a linear map

$$f\colon (E \otimes F) \otimes G \to E \otimes F \otimes G,$$

such that $f((u \otimes v) \otimes w) = u \otimes v \otimes w$.

Also consider the map

$$(u,\, v, w) \mapsto (u \otimes v) \otimes w$$

from $E \times F \times G$ to $(E \otimes F) \otimes G$. It is trilinear, and thus, there is a linear map

$$g\colon E \otimes F \otimes G \to (E \otimes F) \otimes G,$$

such that $g(u \otimes v \otimes w) = (u \otimes v) \otimes w$. Clearly, $f \circ g$ and $g \circ f$ are identity maps, and thus, f and g are isomorphisms. The other cases are similar. $\square$

Given any three vector spaces, E, F, G, we have the canonical isomorphism

$$\mathrm{Hom}(E, F; G) \cong \mathrm{Hom}(E, \mathrm{Hom}(F, G)).$$

Indeed, any bilinear map, $f\colon E \times F \to G$, gives the linear map, $\varphi(f) \in \mathrm{Hom}(E, \mathrm{Hom}(F, G))$, where $\varphi(f)(u)$ is the linear map in $\mathrm{Hom}(F, G)$ given by

$$\varphi(f)(u)(v) = f(u, v).$$

Conversely, given a linear map, $g \in \mathrm{Hom}(E, \mathrm{Hom}(F, G))$, we get the bilinear map, $\psi(g)$, given by

$$\psi(g)(u, v) = g(u)(v),$$

and it is clear that φ and ψ and mutual inverses. Consequently, we have the important corollary:

Proposition 22.8 *For any three vector spaces, E, F, G, we have the canonical isomorphism,*

$$\mathrm{Hom}(E \otimes F, G) \cong \mathrm{Hom}(E, \mathrm{Hom}(F, G)),$$

22.4 Duality for Tensor Products

In this section, all vector spaces are assumed to have finite dimension. Let us now see how tensor products behave under duality. For this, we define a pairing between $E_1^* \otimes \cdots \otimes E_n^*$ and $E_1 \otimes \cdots \otimes E_n$ as follows: For any fixed $(v_1^*, \ldots, v_n^*) \in E_1^* \times \cdots \times E_n^*$, we have the multilinear map,

$$l_{v_1^*, \ldots, v_n^*}\colon (u_1, \ldots, u_n) \mapsto v_1^*(u_1) \cdots v_n^*(u_n),$$

from $E_1 \times \cdots \times E_n$ to K. The map $l_{v_1^*, \ldots, v_n^*}$ extends uniquely to a linear map, $L_{v_1^*, \ldots, v_n^*}\colon E_1 \otimes \cdots \otimes E_n \longrightarrow K$. We also have the multilinear map,

$$(v_1^*, \ldots, v_n^*) \mapsto L_{v_1^*, \ldots, v_n^*},$$

from $E_1^* \times \cdots \times E_n^*$ to $\mathrm{Hom}(E_1 \otimes \cdots \otimes E_n, K)$, which extends to a linear map, L, from $E_1^* \otimes \cdots \otimes E_n^*$ to $\mathrm{Hom}(E_1 \otimes \cdots \otimes E_n, K)$. However, in view of the isomorphism,

$$\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(U, \mathrm{Hom}(V, W)),$$

we can view L as a linear map,

$$L\colon (E_1^* \otimes \cdots \otimes E_n^*) \otimes (E_1 \otimes \cdots \otimes E_n) \to K,$$

which corresponds to a bilinear map,

$$(E_1^* \otimes \cdots \otimes E_n^*) \times (E_1 \otimes \cdots \otimes E_n) \longrightarrow K,$$

via the isomorphism $(U \otimes V)^* \cong \mathrm{L}(U, V; K)$. It is easy to check that this bilinear map is nondegenerate and thus, by Proposition 22.1, we have a canonical isomorphism,

$$(E_1 \otimes \cdots \otimes E_n)^* \cong E_1^* \otimes \cdots \otimes E_n^*.$$

This, together with the isomorphism, $\mathrm{L}(E_1, \ldots, E_n; K) \cong (E_1 \otimes \cdots \otimes E_n)^*$, yields a canonical isomorphism

$$\mathrm{L}(E_1, \ldots, E_n; K) \cong E_1^* \otimes \cdots \otimes E_n^*.$$

We prove another useful canonical isomorphism that allows us to treat linear maps as tensors.

Let E and F be two vector spaces and let $\alpha\colon E^* \times F \to \mathrm{Hom}(E, F)$ be the map defined such that

$$\alpha(u^*, f)(x) = u^*(x)f,$$

for all $u^* \in E^*$, $f \in F$, and $x \in E$. This map is clearly bilinear and thus, it induces a linear map,

$$\alpha_\otimes\colon E^* \otimes F \to \mathrm{Hom}(E, F),$$

such that

$$\alpha_\otimes(u^* \otimes f)(x) = u^*(x)f.$$

Proposition 22.9 *If E and F are vector spaces with E of finite dimension, then the linear map, $\alpha_\otimes\colon E^* \otimes F \to \mathrm{Hom}(E, F)$, is a canonical isomorphism.*

Proof. Let $(e_j)_{1 \leq j \leq n}$ be a basis of E and, as usual, let $e_j^* \in E^*$ be the linear form defined by

$$e_j^*(e_k) = \delta_{j,k},$$

where $\delta_{j,k} = 1$ iff $j = k$ and 0 otherwise. We know that $(e_j^*)_{1 \leq j \leq n}$ is a basis of E^* (this is where we use the finite dimension of E). Now, for any linear map, $f \in \mathrm{Hom}(E, F)$, for every $x = x_1 e_1 + \cdots + x_n e_n \in E$, we have

$$f(x) = f(x_1 e_1 + \cdots + x_n e_n) = x_1 f(e_1) + \cdots + x_n f(e_n) = e_1^*(x) f(e_1) + \cdots + e_n^*(x) f(e_n).$$

Consequently, every linear map, $f \in \mathrm{Hom}(E, F)$, can be expressed as

$$f(x) = e_1^*(x) f_1 + \cdots + e_n^*(x) f_n,$$

for some $f_i \in F$. Furthermore, if we apply f to e_i, we get $f(e_i) = f_i$, so the f_i are unique. Observe that

$$(\alpha_\otimes(e_1^* \otimes f_1 + \cdots + e_n^* \otimes f_n))(x) = \sum_{i=1}^{n} (\alpha_\otimes(e_i^* \otimes f_i))(x) = \sum_{i=1}^{n} e_i^*(x) f_i.$$

Thus, $\alpha_\otimes$ is surjective. As $(e_j^*)_{1\leq j\leq n}$ is a basis of E^*, the tensors $e_j^* \otimes f$, with $f \in F$, span $E^* \otimes F$. Thus, every element of $E^* \otimes F$ is of the form $\sum_{i=1}^n e_i^* \otimes f_i$, for some $f_i \in F$. Assume

$$\alpha_\otimes(\sum_{i=1}^n e_i^* \otimes f_i) = \alpha_\otimes(\sum_{i=1}^n e_i^* \otimes f_i') = f,$$

for some $f_i, f_i' \in F$ and some $f \in \mathrm{Hom}(E, F)$. Then for every $x \in E$,

$$\sum_{i=1}^n e_i^*(x) f_i = \sum_{i=1}^n e_i^*(x) f_i' = f(x).$$

Since the f_i and f_i' are uniquely determined by the linear map, f, we must have $f_i = f_i'$ and $\alpha_\otimes$ is injective. Therefore, $\alpha_\otimes$ is a bijection. $\square$

Note that in Proposition 22.9, the space F may have infinite dimension but E has finite dimension. In view of the canonical isomorphism

$$\mathrm{Hom}(E_1, \ldots, E_n; F) \cong \mathrm{Hom}(E_1 \otimes \cdots \otimes E_n, F)$$

and the canonical isomorphism $(E_1 \otimes \cdots \otimes E_n)^* \cong E_1^* \otimes \cdots \otimes E_n^*$, where the E_i's are finite-dimensional, Proposition 22.9 yields the canonical isomorphism

$$\mathrm{Hom}(E_1, \ldots, E_n; F) \cong E_1^* \otimes \cdots \otimes E_n^* \otimes F.$$

22.5 Tensor Algebras

The tensor product

$$\underbrace{V \otimes \cdots \otimes V}_{m}$$

is also denoted as

$$\bigotimes^m V \quad \text{or} \quad V^{\otimes m}$$

and is called the *m-th tensor power of* V (with $V^{\otimes 1} = V$, and $V^{\otimes 0} = K$). We can pack all the tensor powers of V into the "big" vector space,

$$T(V) = \bigoplus_{m\geq 0} V^{\otimes m},$$

also denoted $T^\bullet(V)$, to avoid confusion with the tangent bundle. This is an interesting object because we can define a multiplication operation on it which makes it into an *algebra* called the *tensor algebra of* V. When V is of finite dimension n, this space corresponds to the algebra of polynomials with coefficients in K in n noncommuting variables.

Let us recall the definition of an algebra over a field. Let K denote any (commutative) field, although for our purposes, we may assume that $K = \mathbb{R}$ (and occasionally, $K = \mathbb{C}$). Since we will only be dealing with associative algebras with a multiplicative unit, we only define algebras of this kind.

Definition 22.3 Given a field, K, a K*-algebra* is a K-vector space, A, together with a bilinear operation, $\cdot\colon A \times A \to A$, called *multiplication*, which makes A into a ring with unity, 1 (or 1_A, when we want to be very precise). This means that $\cdot$ is associative and that there is a multiplicative identity element, 1, so that $1 \cdot a = a \cdot 1 = a$, for all $a \in A$. Given two K-algebras A and B, a K*-algebra homomorphism,* $h\colon A \to B$, is a linear map that is also a ring homomorphism, with $h(1_A) = 1_B$.

For example, the ring, $M_n(K)$, of all $n \times n$ matrices over a field, K, is a K-algebra.

There is an obvious notion of *ideal* of a K-algebra: An ideal, $\mathfrak{A} \subseteq A$, is a linear subspace of A that is also a two-sided ideal with respect to multiplication in A. If the field K is understood, we usually simply say an algebra instead of a K-algebra.

We would like to define a multiplication operation on $T(V)$ which makes it into a K-algebra. As

$$T(V) = \bigoplus_{i\geq 0} V^{\otimes i},$$

for every $i \geq 0$, there is a natural injection $\iota_n\colon V^{\otimes n} \to T(V)$, and in particular, an injection $\iota_0\colon K \to T(V)$. The multiplicative unit, $\mathbf{1}$, of $T(V)$ is the image, $\iota_0(1)$, in $T(V)$ of the unit, 1, of the field K. Since every $v \in T(V)$ can be expressed as a finite sum

$$v = \iota_{n_1}(v_1) + \cdots + \iota_{n_k}(v_k),$$

where $v_i \in V^{\otimes n_i}$ and the n_i are natural numbers with $n_i \neq n_j$ if $i \neq j$, to define multiplication in $T(V)$, using bilinearity, it is enough to define multiplication operations, $\cdot\colon V^{\otimes m} \times V^{\otimes n} \longrightarrow V^{\otimes(m+n)}$, which, using the isomorphisms, $V^{\otimes n} \cong \iota_n(V^{\otimes n})$, yield multiplication operations, $\cdot\colon \iota_m(V^{\otimes m}) \times \iota_n(V^{\otimes n}) \longrightarrow \iota_{m+n}(V^{\otimes(m+n)})$. More precisely, we use the canonical isomorphism,

$$V^{\otimes m} \otimes V^{\otimes n} \cong V^{\otimes(m+n)},$$

which defines a bilinear operation,

$$V^{\otimes m} \times V^{\otimes n} \longrightarrow V^{\otimes(m+n)},$$

which is taken as the multiplication operation. The isomorphism $V^{\otimes m} \otimes V^{\otimes n} \cong V^{\otimes(m+n)}$ can be established by proving the isomorphisms

$$\begin{aligned} V^{\otimes m} \otimes V^{\otimes n} &\cong V^{\otimes m} \otimes \underbrace{V \otimes \cdots \otimes V}_{n} \\ V^{\otimes m} \otimes \underbrace{V \otimes \cdots \otimes V}_{n} &\cong V^{\otimes(m+n)}, \end{aligned}$$

which can be shown using methods similar to those used to proved associativity. Of course, the multiplication, $V^{\otimes m} \times V^{\otimes n} \longrightarrow V^{\otimes(m+n)}$, is defined so that

$$(v_1 \otimes \cdots \otimes v_m) \cdot (w_1 \otimes \cdots \otimes w_n) = v_1 \otimes \cdots \otimes v_m \otimes w_1 \otimes \cdots \otimes w_n.$$

(This has to be made rigorous by using isomorphisms involving the associativity of tensor products, for details, see see Atiyah and Macdonald [9].)

Remark: It is important to note that multiplication in $T(V)$ is **not** commutative. Also, in all rigor, the unit, $\mathbf{1}$, of $T(V)$ is **not equal** to 1, the unit of the field K. However, in view of the injection $\iota_0\colon K \to T(V)$, for the sake of notational simplicity, we will denote $\mathbf{1}$ by 1. More generally, in view of the injections $\iota_n\colon V^{\otimes n} \to T(V)$, we identify elements of $V^{\otimes n}$ with their images in $T(V)$.

The algebra, $T(V)$, satisfies a universal mapping property which shows that it is unique up to isomorphism. For simplicity of notation, let $i\colon V \to T(V)$ be the natural injection of V into $T(V)$.

Proposition 22.10 *Given any K-algebra, A, for any linear map, $f\colon V \to A$, there is a unique K-algebra homomorphism, $\overline{f}\colon T(V) \to A$, so that*

$$f = \overline{f} \circ i,$$

as in the diagram below:

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "f"'] & T(V) \arrow[d, "\overline{f}"] \\ & A \end{tikzcd}$$

Proof. Left an an exercise (use Theorem 22.5). $\square$

Most algebras of interest arise as well-chosen quotients of the tensor algebra $T(V)$. This is true for the *exterior algebra,* $\bigwedge(V)$ (also called *Grassmann algebra*), where we take the quotient of $T(V)$ modulo the ideal generated by all elements of the form $v \otimes v$, where $v \in V$, and for the *symmetric algebra,* $\mathrm{Sym}(V)$, where we take the quotient of $T(V)$ modulo the ideal generated by all elements of the form $v \otimes w - w \otimes v$, where $v, w \in V$.

Algebras such as $T(V)$ are graded, in the sense that there is a sequence of subspaces, $V^{\otimes n} \subseteq T(V)$, such that

$$T(V) = \bigoplus_{k \geq 0} V^{\otimes n}$$

and the multiplication, $\otimes$, behaves well w.r.t. the grading, *i.e.*, $\otimes\colon V^{\otimes m} \times V^{\otimes n} \to V^{\otimes (m+n)}$. Generally, a K-algebra, E, is said to be a *graded algebra* iff there is a sequence of subspaces, $E^n \subseteq E$, such that

$$E = \bigoplus_{k \geq 0} E^n$$

($E^0 = K$) and the multiplication, $\cdot$, respects the grading, that is, $\cdot\colon E^m \times E^n \to E^{m+n}$. Elements in E^n are called *homogeneous elements of rank (or degree) n.*

In differential geometry and in physics it is necessary to consider slightly more general tensors.

Definition 22.4 Given a vector space, V, for any pair of nonnegative integers, (r, s), the *tensor space, $T^{r,s}(V)$, of type* (r, s), is the tensor product

$$T^{r,s}(V) = V^{\otimes r} \otimes (V^*)^{\otimes s} = \underbrace{V \otimes \cdots \otimes V}_{r} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s},$$

with $T^{0,0}(V) = K$. We also define the *tensor algebra*, $T^{\bullet,\bullet}(V)$, as the coproduct

$$T^{\bullet,\bullet}(V) = \bigoplus_{r,s \geq 0} T^{r,s}(V).$$

Tensors in $T^{r,s}(V)$ are called *homogeneous of degree* (r, s).

Note that tensors in $T^{r,0}(V)$ are just our "old tensors" in $V^{\otimes r}$. We make $T^{\bullet,\bullet}(V)$ into an algebra by defining multiplication operations,

$$T^{r_1,s_1}(V) \times T^{r_2,s_2}(V) \longrightarrow T^{r_1+r_2,s_1+s_2}(V),$$

in the usual way, namely: For $u = u_1 \otimes \cdots \otimes u_{r_1} \otimes u_1^* \otimes \cdots \otimes u_{s_1}^*$ and $v = v_1 \otimes \cdots \otimes v_{r_2} \otimes v_1^* \otimes \cdots \otimes v_{s_2}^*$, let

$$u \otimes v = u_1 \otimes \cdots \otimes u_{r_1} \otimes v_1 \otimes \cdots \otimes v_{r_2} \otimes u_1^* \otimes \cdots \otimes u_{s_1}^* \otimes v_1^* \otimes \cdots \otimes v_{s_2}^*.$$

Denote by $\mathrm{Hom}(V^r, (V^*)^s; W)$ the vector space of all multilinear maps from $V^r \times (V^*)^s$ to W. Then, we have the universal mapping property which asserts that there is a canonical isomorphism

$$\mathrm{Hom}(T^{r,s}(V), W) \cong \mathrm{Hom}(V^r, (V^*)^s; W).$$

In particular,

$$(T^{r,s}(V))^* \cong \mathrm{Hom}(V^r, (V^*)^s; K).$$

For finite dimensional vector spaces, the duality of Section 22.4 is also easily extended to the tensor spaces $T^{r,s}(V)$. We define the pairing

$$T^{r,s}(V^*) \times T^{r,s}(V) \longrightarrow K$$

as follows: If

$$v^* = v_1^* \otimes \cdots \otimes v_r^* \otimes u_{r+1} \otimes \cdots \otimes u_{r+s} \in T^{r,s}(V^*)$$

and

$$u = u_1 \otimes \cdots \otimes u_r \otimes v_{r+1}^* \otimes \cdots \otimes v_{r+s}^* \in T^{r,s}(V),$$

then

$$(v^*, u) = v_1^*(u_1) \cdots v_{r+s}^*(u_{r+s}).$$

This is a nondegenerate pairing and thus, we get a canonical isomorphism,

$$(T^{r,s}(V))^* \cong T^{r,s}(V^*).$$

Consequently, we get a canonical isomorphism,

$$T^{r,s}(V^*) \cong \mathrm{Hom}(V^r, (V^*)^s; K).$$

Remark: The tensor spaces, $T^{r,s}(V)$ are also denoted $T^r_s(V)$. A tensor, $\alpha \in T^{r,s}(V)$ is said to be *contravariant* in the first r arguments and *covariant* in the last s arguments. This terminology refers to the way tensors behave under coordinate changes. Given a basis, $(e_1, \ldots, e_n)$, of V, if $(e^*_1, \ldots, e^*_n)$ denotes the dual basis, then every tensor $\alpha \in T^{r,s}(V)$ is given by an expression of the form

$$\alpha = \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots,j_s}} a^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^*_{j_1} \otimes \cdots \otimes e^*_{j_s}.$$

The tradition in classical tensor notation is to use lower indices on vectors and upper indices on linear forms and in accordance to *Einstein summation convention* (or *Einstein notation*) the position of the indices on the coefficients is reversed. *Einstein summation convention* is to assume that a summation is performed for all values of every index that appears simultaneously once as an upper index and once as a lower index. According to this convention, the tensor α above is written

$$\alpha = a^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}.$$

An older view of tensors is that they are multidimensional arrays of coefficients,

$$\left(a^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}\right),$$

subject to the rules for changes of bases.

Another operation on general tensors, contraction, is useful in differential geometry.

Definition 22.5 For all $r, s \geq 1$, the *contraction*, $c_{i,j} \colon T^{r,s}(V) \to T^{r-1,s-1}(V)$, with $1 \leq i \leq r$ and $1 \leq j \leq s$, is the linear map defined on generators by

$$\begin{aligned} & c_{i,j}(u_1 \otimes \cdots \otimes u_r \otimes v^*_1 \otimes \cdots \otimes v^*_s) \\ & \qquad = v^*_j(u_i)\, u_1 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_r \otimes v^*_1 \otimes \cdots \otimes \widehat{v^*_j} \otimes \cdots \otimes v^*_s, \end{aligned}$$

where the hat over an argument means that it should be omitted.

Let us figure our what is $c_{1,1} \colon T^{1,1}(V) \to \mathbb{R}$, that is $c_{1,1} \colon V \otimes V^* \to \mathbb{R}$. If $(e_1, \ldots, e_n)$ is a basis of V and $(e^*_1, \ldots, e^*_n)$ is the dual basis, every $h \in V \otimes V^* \cong \mathrm{Hom}(V, V)$ can be expressed as

$$h = \sum_{i,j=1}^{n} a_{ij}\, e_i \otimes e^*_j.$$

As
$$c_{1,1}(e_i \otimes e_j^*) = \delta_{i,j},$$
we get
$$c_{1,1}(h) = \sum_{i=1}^{n} a_{ii} = \mathrm{tr}(h),$$
where $\mathrm{tr}(h)$ is the *trace* of h, where h is viewed as the linear map given by the matrix, (a_{ij}). Actually, since $c_{1,1}$ is defined independently of any basis, $c_{1,1}$ provides an intrinsic definition of the trace of a linear map, $h \in \mathrm{Hom}(V, V)$.

Remark: Using the Einstein summation convention, if
$$\alpha = a^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s},$$
then
$$c_{k,l}(\alpha) = a^{i_1,\ldots,i_{k-1},i,i_{k+1}\ldots,i_r}_{j_1,\ldots,j_{l-1},i,j_{l+1},\ldots,j_s} e_{i_1} \otimes \cdots \otimes \widehat{e_{i_k}} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes \widehat{e^{j_l}} \otimes \cdots \otimes e^{j_s}.$$

If E and F are two K-algebras, we know that their tensor product, $E \otimes F$, exists as a vector space. We can make $E \otimes F$ into an algebra as well. Indeed, we have the multilinear map
$$E \times F \times E \times F \longrightarrow E \otimes F$$
given by $(a, b, c, d) \mapsto (ac) \otimes (bd)$, where ac is the product of a and c in E and bd is the product of b and d in F. By the universal mapping property, we get a linear map,
$$E \otimes F \otimes E \otimes F \longrightarrow E \otimes F.$$
Using the isomorphism,
$$E \otimes F \otimes E \otimes F \cong (E \otimes F) \otimes (E \otimes F),$$
we get a linear map,
$$(E \otimes F) \otimes (E \otimes F) \longrightarrow E \otimes F,$$
and thus, a bilinear map,
$$(E \otimes F) \times (E \otimes F) \longrightarrow E \otimes F,$$
which is our multiplication operation in $E \otimes F$. This multiplication is determined by
$$(a \otimes b) \cdot (c \otimes d) = (ac) \otimes (bd).$$
One immediately checks that $E \otimes F$ with this multiplication is a K-algebra.

We now turn to symmetric tensors.

22.6 Symmetric Tensor Powers

Our goal is to come up with a notion of tensor product that will allow us to treat symmetric multilinear maps as linear maps. First, note that we have to restrict ourselves to a single vector space, E, rather then n vector spaces $E_1, \ldots, E_n$, so that symmetry makes sense. Recall that a multilinear map, $f\colon E^n \to F$, is *symmetric* iff

$$f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) = f(u_1, \ldots, u_n),$$

for all $u_i \in E$ and all permutations, $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$. The group of permutations on $\{1, \ldots, n\}$ (the *symmetric group*) is denoted $\mathfrak{S}_n$. The vector space of all symmetric multilinear maps, $f\colon E^n \to F$, is denoted by $\mathrm{S}^n(E; F)$. Note that $\mathrm{S}^1(E; F) = \mathrm{Hom}(E, F)$.

We could proceed directly as in Theorem 22.5, and construct symmetric tensor products from scratch. However, since we already have the notion of a tensor product, there is a more economical method. First, we define symmetric tensor powers.

Definition 22.6 An *n-th symmetric tensor power* of a vector space E, where $n \geq 1$, is a vector space S, together with a symmetric multilinear map $\varphi\colon E^n \to S$, such that, for every vector space F and for every symmetric multilinear map $f\colon E^n \to F$, there is a unique linear map $f_\odot\colon S \to F$, with

$$f(u_1, \ldots, u_n) = f_\odot(\varphi(u_1, \ldots, u_n)),$$

for all $u_1, \ldots, u_n \in E$, or for short

$$f = f_\odot \circ \varphi.$$

Equivalently, there is a unique linear map $f_\odot$ such that the following diagram commutes:

$$\begin{tikzcd} E^n \arrow[r, "\varphi"] \arrow[dr, "f"'] & S \arrow[d, "f_\odot"] \\ & F \end{tikzcd}$$

First, we show that any two symmetric n-th tensor powers (S_1, φ_1) and (S_2, φ_2) for E, are isomorphic.

Proposition 22.11 *Given any two symmetric n-th tensor powers (S_1, φ_1) and (S_2, φ_2) for E, there is an isomorphism $h\colon S_1 \to S_2$ such that*

$$\varphi_2 = h \circ \varphi_1.$$

Proof. Replace tensor product by n-th symmetric tensor power in the proof of Proposition 22.4. $\square$

We now give a construction that produces a symmetric n-th tensor power of a vector space E.

Theorem 22.12 *Given a vector space E, a symmetric n-th tensor power $(\mathrm{Sym}^n(E), \varphi)$ for E can be constructed ($n \geq 1$). Furthermore, denoting $\varphi(u_1, \ldots, u_n)$ as $u_1 \odot \cdots \odot u_n$, the symmetric tensor power $\mathrm{Sym}^n(E)$ is generated by the vectors $u_1 \odot \cdots \odot u_n$, where $u_1, \ldots, u_n \in E$, and for every symmetric multilinear map $f \colon E^n \to F$, the unique linear map $f_\odot \colon \mathrm{Sym}^n(E) \to F$ such that $f = f_\odot \circ \varphi$, is defined by*

$$f_\odot(u_1 \odot \cdots \odot u_n) = f(u_1, \ldots, u_n),$$

on the generators $u_1 \odot \cdots \odot u_n$ of $\mathrm{Sym}^n(E)$.

Proof. The tensor power $E^{\otimes n}$ is too big, and thus, we define an appropriate quotient. Let C be the subspace of $E^{\otimes n}$ generated by the vectors of the form

$$u_1 \otimes \cdots \otimes u_n - u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)},$$

for all $u_i \in E$, and all permutations $\sigma \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$. We claim that the quotient space $(E^{\otimes n})/C$ does the job.

Let $p \colon E^{\otimes n} \to (E^{\otimes n})/C$ be the quotient map. Let $\varphi \colon E^n \to (E^{\otimes n})/C$ be the map

$$(u_1, \ldots, u_n) \mapsto p(u_1 \otimes \cdots \otimes u_n),$$

or equivalently, $\varphi = p \circ \varphi_0$, where $\varphi_0(u_1, \ldots, u_n) = u_1 \otimes \cdots \otimes u_n$.

Let us denote $\varphi(u_1, \ldots, u_n)$ as $u_1 \odot \cdots \odot u_n$. It is clear that φ is symmetric. Since the vectors $u_1 \otimes \cdots \otimes u_n$ generate $E^{\otimes n}$, and p is surjective, the vectors $u_1 \odot \cdots \odot u_n$ generate $(E^{\otimes n})/C$.

Given any symmetric multilinear map $f \colon E^n \to F$, there is a linear map $f_\otimes \colon E^{\otimes n} \to F$ such that $f = f_\otimes \circ \varphi_0$, as in the diagram below:

$$\begin{tikzcd} E^n \arrow[r, "\varphi_0"] \arrow[dr, "f"'] & E^{\otimes n} \arrow[d, "f_\otimes"] \\ & F \end{tikzcd}$$

However, since f is symmetric, we have $f_\otimes(z) = 0$ for every $z \in E^{\otimes n}$. Thus, we get an induced linear map $h \colon (E^{\otimes n})/C \to F$, such that $h([z]) = f_\otimes(z)$, where $[z]$ is the equivalence class in $(E^{\otimes n})/C$ of $z \in E^{\otimes n}$:

$$\begin{tikzcd} E^n \arrow[r, "p \circ \varphi_0"] \arrow[dr, "f"'] & (E^{\otimes n})/C \arrow[d, "h"] \\ & F \end{tikzcd}$$

However, if a linear map $f_\odot \colon (E^{\otimes n})/C \to F$ exists, since the vectors $u_1 \odot \cdots \odot u_n$ generate $(E^{\otimes n})/C$, we must have

$$f_\odot(u_1 \odot \cdots \odot u_n) = f(u_1, \ldots, u_n),$$

which shows that h and $f_{\odot}$ agree. Thus, $\mathrm{Sym}^n(E) = (E^{\otimes n})/C$ and φ constitute a symmetric n-th tensor power of E. $\square$

Again, the actual construction is not important. What is important is that the symmetric n-th power has the universal mapping property with respect to symmetric multilinear maps.

Remark: The notation $\odot$ for the commutative multiplication of symmetric tensor powers is not standard. Another notation commonly used is $\cdot$. We often abbreviate "symmetric tensor power" as "symmetric power". The symmetric power, $\mathrm{Sym}^n(E)$, is also denoted $\mathrm{Sym}^n E$ or $S(E)$. To be consistent with the use of $\odot$, we could have used the notation $\bigodot^n E$. Clearly, $\mathrm{Sym}^1(E) \cong E$ and it is convenient to set $\mathrm{Sym}^0(E) = K$.

The fact that the map $\varphi\colon E^n \to \mathrm{Sym}^n(E)$ is symmetric and multinear, can also be expressed as follows:

$$\begin{aligned} u_1 \odot \cdots \odot (u_i + v_i) \odot \cdots \odot u_n &= (u_1 \odot \cdots \odot u_i \odot \cdots \odot u_n) \\ &\quad + (u_1 \odot \cdots \odot v_i \odot \cdots \odot u_n), \\ u_1 \odot \cdots \odot (\lambda u_i) \odot \cdots \odot u_n &= \lambda(u_1 \odot \cdots \odot u_i \odot \cdots \odot u_n), \\ u_{\sigma(1)} \odot \cdots \odot u_{\sigma(n)} &= u_1 \odot \cdots \odot u_n, \end{aligned}$$

for all permutations $\sigma \in \mathfrak{S}_n$.

The last identity shows that the "operation" $\odot$ is commutative. Thus, we can view the symmetric tensor $u_1 \odot \cdots \odot u_n$ as a multiset.

Theorem 22.12 yields a canonical isomorphism

$$\mathrm{Hom}(\mathrm{Sym}^n(E), F) \cong \mathrm{S}(E^n; F),$$

between the vector space of linear maps $\mathrm{Hom}(\mathrm{Sym}^n(E), F)$, and the vector space of symmetric multilinear maps $\mathrm{S}(E^n; F)$, via the linear map $- \circ \varphi$ defined by

$$h \mapsto h \circ \varphi,$$

where $h \in \mathrm{Hom}(\mathrm{Sym}^n(E), F)$. Indeed, $h \circ \varphi$ is clearly symmetric multilinear, and since by Theorem 22.12, for every symmetric multilinear map $f \in \mathrm{S}(E^n; F)$, there is a unique linear map $f_{\odot} \in \mathrm{Hom}(\mathrm{Sym}^n(E), F)$ such that $f = f_{\odot} \circ \varphi$, the map $- \circ \varphi$ is bijective. As a matter of fact, its inverse is the map

$$f \mapsto f_{\odot}.$$

In particular, when $F = K$, we get a canonical isomorphism

$$(\mathrm{Sym}^n(E))^* \cong \mathrm{S}^n(E; K).$$

Symmetric tensors in $\mathrm{Sym}^n(E)$ are also called *symmetric n-tensors*, and tensors of the form $u_1 \odot \cdots \odot u_n$, where $u_i \in E$, are called *simple (or decomposable) symmetric n-tensors*.

Those symmetric n-tensors that are not simple are often called *compound symmetric n-tensors*.

Given two linear maps $f\colon E \to E'$ and $g\colon E \to E'$, we can define $h\colon E \times E \to \mathrm{Sym}^2(E')$ by

$$h(u, v) = f(u) \odot g(v).$$

It is immediately verified that h is symmetric bilinear, and thus, it induces a unique linear map

$$f \odot g\colon \mathrm{Sym}^2(E) \to \mathrm{Sym}^2(E'),$$

such that

$$(f \odot g)(u \odot v) = f(u) \odot g(u).$$

If we also have linear maps $f'\colon E' \to E''$ and $g'\colon E' \to E''$, we can easily verify that

$$(f' \circ f) \odot (g' \circ g) = (f' \odot g') \circ (f \odot g).$$

The generalization to the symmetric tensor product $f_1 \odot \cdots \odot f_n$ of $n \geq 3$ linear maps $f_i\colon E \to E'$ is immediate, and left to the reader.

22.7 Bases of Symmetric Powers

The vectors $u_1 \odot \cdots \odot u_n$, where $u_1, \ldots, u_n \in E$, generate $\mathrm{Sym}^n(E)$, but they are not linearly independent. We will prove a version of Proposition 22.6 for symmetric tensor powers. For this, recall that a (finite) multiset over a set I is a function $M\colon I \to \mathbb{N}$, such that $M(i) \neq 0$ for finitely many $i \in I$, and that the set of all multisets over I is denoted as $\mathbb{N}^{(I)}$. We let $\mathrm{dom}(M) = \{i \in I \mid M(i) \neq 0\}$, which is a finite set. Then, for any multiset $M \in \mathbb{N}^{(I)}$, note that the sum $\sum_{i\in I} M(i)$ makes sense, since $\sum_{i\in I} M(i) = \sum_{i\in \mathrm{dom}(M)} M(i)$, and $\mathrm{dom}(M)$ is finite. For every multiset $M \in \mathbb{N}^{(I)}$, for any $n \geq 2$, we define the set J_M of functions $\eta\colon \{1, \ldots, n\} \to \mathrm{dom}(M)$, as follows:

$$J_M = \{\eta \mid \eta\colon \{1, \ldots, n\} \to \mathrm{dom}(M),\ |\eta^{-1}(i)| = M(i),\ i \in \mathrm{dom}(M),\ \sum_{i\in I} M(i) = n\}.$$

In other words, if $\sum_{i\in I} M(i) = n$ and $\mathrm{dom}(M) = \{i_1, \ldots, i_k\}$,[1] any function $\eta \in J_M$ specifies a sequence of length n, consisting of $M(i_1)$ occurrences of i_1, $M(i_2)$ occurrences of $i_2, \ldots,$ $M(i_k)$ occurrences of i_k. Intuitively, any η defines a “permutation” of the sequence (of length n)

$$\langle \underbrace{i_1, \ldots, i_1}_{M(i_1)}, \underbrace{i_2, \ldots, i_2}_{M(i_2)}, \ldots, \underbrace{i_k, \ldots, i_k}_{M(i_k)} \rangle.$$

[1] Note that must have $k \leq n$.

Given any $k \geq 1$, and any $u \in E$, we denote

$$\underbrace{u \odot \cdots \odot u}_{k}$$

as $u^{\odot k}$.

We can now prove the following Proposition.

Proposition 22.13 *Given a vector space E, if $(u_i)_{i\in I}$ is a basis for E, then the family of vectors*

$$\left(u_{i_1}^{\odot M(i_1)} \odot \cdots \odot u_{i_k}^{\odot M(i_k)}\right)_{M\in\mathbb{N}^{(I)},\, \sum_{i\in I} M(i)=n,\, \{i_1,\ldots,i_k\}=\mathrm{dom}(M)}$$

is a basis of the symmetric n-th tensor power $\mathrm{Sym}^n(E)$.

Proof. The proof is very similar to that of Proposition 22.6. For any nontrivial vector space F, for any family of vectors

$$(w_M)_{M\in\mathbb{N}^{(I)},\, \sum_{i\in I} M(i)=n},$$

we show the existence of a symmetric multilinear map $h\colon \mathrm{Sym}^n(E) \to F$, such that for every $M \in \mathbb{N}^{(I)}$ with $\sum_{i\in I} M(i) = n$, we have

$$h(u_{i_1}^{\odot M(i_1)} \odot \cdots \odot u_{i_k}^{\odot M(i_k)}) = w_M,$$

where $\{i_1, \ldots, i_k\} = \mathrm{dom}(M)$. We define the map $f\colon E^n \to F$ as follows:

$$f(\sum_{j_1\in I} v_{j_1}^1 u_{j_1}^1, \ldots, \sum_{j_n\in I} v_{j_n}^n u_{j_n}^n) = \sum_{\substack{M\in\mathbb{N}^{(I)} \\ \sum_{i\in I} M(i)=n}} \left(\sum_{\eta\in J_M} v_{\eta(1)}^1 \cdots v_{\eta(n)}^n\right) w_M.$$

It is not difficult to verify that f is symmetric and multilinear. By the universal mapping property of the symmetric tensor product, the linear map $f_\odot\colon \mathrm{Sym}^n(E) \to F$ such that $f = f_\odot \circ \varphi$, is the desired map h. Then, by Proposition 22.3, it follows that the family

$$\left(u_{i_1}^{\odot M(i_1)} \odot \cdots \odot u_{i_k}^{\odot M(i_k)}\right)_{M\in\mathbb{N}^{(I)},\, \sum_{i\in I} M(i)=n,\, \{i_1,\ldots,i_k\}=\mathrm{dom}(M)}$$

is linearly independent. Using the commutativity of $\odot$, we can also show that these vectors generate $\mathrm{Sym}^n(E)$, and thus, they form a basis for $\mathrm{Sym}^n(E)$. The details are left as an exercise. $\square$

As a consequence, when I is finite, say of size $p = \dim(E)$, the dimension of $\mathrm{Sym}^n(E)$ is the number of finite multisets $(j_1, \ldots, j_p)$, such that $j_1 + \cdots + j_p = n$, $j_k \geq 0$. We leave as an exercise to show that this number is $\binom{p+n-1}{n}$. Thus, if $\dim(E) = p$, then the dimension of

$\mathrm{Sym}^n(E)$ is $\binom{p+n-1}{n}$. Compare with the dimension of $E^{\otimes n}$, which is p^n. In particular, when $p = 2$, the dimension of $\mathrm{Sym}^n(E)$ is $n + 1$. This can also be seen directly.

Remark: The number $\binom{p+n-1}{n}$ is also the number of homogeneous monomials

$$X_1^{j_1} \cdots X_p^{j_p}$$

of total degree n in p variables (we have $j_1 + \cdots + j_p = n$). This is not a coincidence! Symmetric tensor products are closely related to polynomials (for more on this, see the next remark).

Given a vector space E and a basis $(u_i)_{i\in I}$ for E, Proposition 22.13 shows that every symmetric tensor $z \in \mathrm{Sym}^n(E)$ can be written in a unique way as

$$z = \sum_{\substack{M\in\mathbb{N}^{(I)}\\ \sum_{i\in I} M(i)=n\\ \{i_1,\ldots,i_k\}=\mathrm{dom}(M)}} \lambda_M \, u_{i_1}^{\odot M(i_1)} \odot \cdots \odot u_{i_k}^{\odot M(i_k)},$$

for some unique family of scalars $\lambda_M \in K$, all zero except for a finite number.

This looks like a homogeneous polynomial of total degree n, where the monomials of total degree n are the symmetric tensors

$$u_{i_1}^{\odot M(i_1)} \odot \cdots \odot u_{i_k}^{\odot M(i_k)},$$

in the "indeterminates" u_i, where $i \in I$ (recall that $M(i_1) + \cdots + M(i_k) = n$). Again, this is not a coincidence. Polynomials can be defined in terms of symmetric tensors.

22.8 Some Useful Isomorphisms for Symmetric Powers

We can show the following property of the symmetric tensor product, using the proof technique of Proposition 22.7:

$$\mathrm{Sym}^n(E \oplus F) \cong \bigoplus_{k=0}^{n} \mathrm{Sym}^k(E) \otimes \mathrm{Sym}^{n-k}(F).$$

22.9 Duality for Symmetric Powers

In this section, all vector spaces are assumed to have finite dimension. We define a nondegenerate pairing, $\mathrm{Sym}^n(E^*) \times \mathrm{Sym}^n(E) \longrightarrow K$, as follows: Consider the multilinear map,

$$(E^*)^n \times E^n \longrightarrow K,$$

given by

$$(v_1^*, \ldots, v_n^*, u_1, \ldots, u_n) \mapsto \sum_{\sigma\in\mathfrak{S}_n} v_{\sigma(1)}^*(u_1) \cdots v_{\sigma(n)}^*(u_n).$$

Note that the expression on the right-hand side is "almost" the determinant, $\det(v_j^*(u_i))$, except that the sign $\mathrm{sgn}(\sigma)$ is missing (where $\mathrm{sgn}(\sigma)$ is the signature of the permutation σ, that is, the parity of the number of transpositions into which σ can be factored). Such an expression is called a *permanent*. It is easily checked that this expression is symmetric w.r.t. the u_i's and also w.r.t. the v_j^*. For any fixed $(v_1^*, \ldots, v_n^*) \in (E^*)^n$, we get a symmetric multinear map,

$$l_{v_1^*,\ldots,v_n^*}\colon (u_1, \ldots, u_n) \mapsto \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)}^*(u_1) \cdots v_{\sigma(n)}^*(u_n),$$

from E^n to K. The map $l_{v_1^*,\ldots,v_n^*}$ extends uniquely to a linear map, $L_{v_1^*,\ldots,v_n^*}\colon \mathrm{Sym}^n(E) \to K$. Now, we also have the symmetric multilinear map,

$$(v_1^*, \ldots, v_n^*) \mapsto L_{v_1^*,\ldots,v_n^*},$$

from $(E^*)^n$ to $\mathrm{Hom}(\mathrm{Sym}^n(E), K)$, which extends to a linear map, L, from $\mathrm{Sym}^n(E^*)$ to $\mathrm{Hom}(\mathrm{Sym}^n(E), K)$. However, in view of the isomorphism,

$$\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(U, \mathrm{Hom}(V, W)),$$

we can view L as a linear map,

$$L\colon \mathrm{Sym}^n(E^*) \otimes \mathrm{Sym}^n(E) \longrightarrow K,$$

which corresponds to a bilinear map,

$$\mathrm{Sym}^n(E^*) \times \mathrm{Sym}^n(E) \longrightarrow K.$$

Now, this pairing in nondegenerate. This can be done using bases and we leave it as an exercise to the reader (see Knapp [89], Appendix A). Therefore, we get a canonical isomorphism,

$$(\mathrm{Sym}^n(E))^* \cong \mathrm{Sym}^n(E^*).$$

Since we also have an isomorphism

$$(\mathrm{Sym}^n(E))^* \cong \mathrm{S}^n(E, K),$$

we get a canonical isomorphism

$$\mathrm{Sym}^n(E^*) \cong \mathrm{S}^n(E, K)$$

which allows us to interpret symmetric tensors over E^* as symmetric multilinear maps.

Remark: The isomorphism, $\mu\colon \mathrm{Sym}^n(E^*) \cong \mathrm{S}^n(E, K)$, discussed above can be described explicity as the linear extension of the map given by

$$\mu(v_1^* \odot \cdots \odot v_n^*)(u_1, \ldots, u_n) = \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)}^*(u_1) \cdots v_{\sigma(n)}^*(u_n).$$

Now, the map from E^n to $\mathrm{Sym}^n(E)$ given by $(u_1, \ldots, u_n) \mapsto u_1 \odot \cdots \odot u_n$ yields a surjection, $\pi\colon E^{\otimes n} \to \mathrm{Sym}^n(E)$. Because we are dealing with vector spaces, this map has some section, that is, there is some injection, $\iota\colon \mathrm{Sym}^n(E) \to E^{\otimes n}$, with $\pi \circ \iota = \mathrm{id}$. If our field, K, has characteristic 0, then there is a special section having a natural definition involving a symmetrization process defined as follows: For every permutation, σ, we have the map, $r_\sigma\colon E^n \to E^{\otimes n}$, given by

$$r_\sigma(u_1, \ldots, u_n) = u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}.$$

As r_σ is clearly multilinear, r_σ extends to a linear map, $r_\sigma\colon E^{\otimes n} \to E^{\otimes n}$, and we get a map, $\mathfrak{S}_n \times E^{\otimes n} \longrightarrow E^{\otimes n}$, namely,

$$\sigma \cdot z = r_\sigma(z).$$

It is immediately checked that this is a left action of the symmetric group, $\mathfrak{S}_n$, on $E^{\otimes n}$ and the tensors $z \in E^{\otimes n}$ such that

$$\sigma \cdot z = z, \quad \text{for all} \quad \sigma \in \mathfrak{S}_n$$

are called *symmetrized* tensors. We define the map, $\iota\colon E^n \to E^{\otimes n}$, by

$$\iota(u_1, \ldots, u_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma \cdot (u_1 \otimes \cdots \otimes u_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}.$$

As the right hand side is clearly symmetric, we get a linear map, $\iota\colon \mathrm{Sym}^n(E) \to E^{\otimes n}$. Clearly, $\iota(\mathrm{Sym}^n(E))$ is the set of symmetrized tensors in $E^{\otimes n}$. If we consider the map, $S = \iota \circ \pi\colon E^{\otimes n} \longrightarrow E^{\otimes n}$, it is easy to check that $S \circ S = S$. Therefore, S is a projection and by linear algebra, we know that

$$E^{\otimes n} = S(E^{\otimes n}) \oplus \mathrm{Ker}\, S = \iota(\mathrm{Sym}^n(E)) \oplus \mathrm{Ker}\, S.$$

It turns out that $\mathrm{Ker}\, S = E^{\otimes n} \cap \mathfrak{I} = \mathrm{Ker}\ \pi$, where $\mathfrak{I}$ is the two-sided ideal of $T(E)$ generated by all tensors of the form $u \otimes v - v \otimes u \in E^{\otimes 2}$ (for example, see Knapp [89], Appendix A). Therefore, ι is injective,

$$E^{\otimes n} = \iota(\mathrm{Sym}^n(E)) \oplus E^{\otimes n} \cap \mathfrak{I} = \iota(\mathrm{Sym}^n(E)) \oplus \mathrm{Ker}\, \pi,$$

and the symmetric tensor power, $\mathrm{Sym}^n(E)$, is naturally embedded into $E^{\otimes n}$.

22.10 Symmetric Algebras

As in the case of tensors, we can pack together all the symmetric powers, $\mathrm{Sym}^n(V)$, into an algebra,

$$\mathrm{Sym}(V) = \bigoplus_{m \geq 0} \mathrm{Sym}^m(V),$$

called the *symmetric tensor algebra of* V. We could adapt what we did in Section 22.5 for general tensor powers to symmetric tensors but since we already have the algebra, $T(V)$, we can proceed faster. If $\mathfrak{I}$ is the two-sided ideal generated by all tensors of the form $u \otimes v - v \otimes u \in V^{\otimes 2}$, we set

$$\mathrm{Sym}^{\bullet}(V) = T(V)/\mathfrak{I}.$$

Then, $\mathrm{Sym}^{\bullet}(V)$ automatically inherits a multiplication operation which is commutative and since $T(V)$ is graded, that is,

$$T(V) = \bigoplus_{m \geq 0} V^{\otimes m},$$

we have

$$\mathrm{Sym}^{\bullet}(V) = \bigoplus_{m \geq 0} V^{\otimes m}/(\mathfrak{I} \cap V^{\otimes m}).$$

However, it is easy to check that

$$\mathrm{Sym}^{m}(V) \cong V^{\otimes m}/(\mathfrak{I} \cap V^{\otimes m}),$$

so

$$\mathrm{Sym}^{\bullet}(V) \cong \mathrm{Sym}(V).$$

When V is of finite dimension, n, $T(V)$ corresponds to the algebra of polynomials with coefficients in K in n variables (this can be seen from Proposition 22.13). When V is of infinite dimension and $(u_i)_{i \in I}$ is a basis of V, the algebra, $\mathrm{Sym}(V)$, corresponds to the algebra of polynomials in infinitely many variables in I. What's nice about the symmetric tensor algebra, $\mathrm{Sym}(V)$, is that it provides an intrinsic definition of a polynomial algebra in any set, I, of variables.

It is also easy to see that $\mathrm{Sym}(V)$ satisfies the following universal mapping property:

Proposition 22.14 *Given any commutative K-algebra, A, for any linear map, $f\colon V \to A$, there is a unique K-algebra homomorphism, $\overline{f}\colon \mathrm{Sym}(V) \to A$, so that*

$$f = \overline{f} \circ i,$$

as in the diagram below:

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "f"'] & \mathrm{Sym}(V) \arrow[d, "\overline{f}"] \\ & A \end{tikzcd}$$

Remark: If E is finite-dimensional, recall the isomorphism, $\mu\colon \mathrm{Sym}^n(E^*) \longrightarrow \mathrm{S}^n(E, K)$, defined as the linear extension of the map given by

$$\mu(v_1^* \odot \cdots \odot v_n^*)(u_1, \ldots, u_n) = \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)}^*(u_1) \cdots v_{\sigma(n)}^*(u_n),$$

Now, we have also a multiplication operation, $\mathrm{Sym}^m(E^*)\times\mathrm{Sym}^n(E^*)\longrightarrow\mathrm{Sym}^{m+n}(E^*)$. The following question then arises:

Can we define a multiplication, $\mathrm{S}^m(E,K)\times\mathrm{S}^n(E,K)\longrightarrow\mathrm{S}^{m+n}(E,K)$, directly on symmetric multilinear forms, so that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Sym}^m(E^*)\times\mathrm{Sym}^n(E^*) \arrow[r, "\odot"] \arrow[d, "\mu\times\mu"] & \mathrm{Sym}^{m+n}(E^*) \arrow[d, "\mu"] \\
\mathrm{S}^m(E,K)\times\mathrm{S}^n(E,K) \arrow[r, "\cdot"] & \mathrm{S}^{m+n}(E,K).
\end{tikzcd}$$

The answer is *yes*! The solution is to define this multiplication such that, for $f\in\mathrm{S}^m(E,K)$ and $g\in\mathrm{S}^n(E,K)$,

$$(f\cdot g)(u_1,\ldots,u_{m+n})=\sum_{\sigma\in\mathrm{shuffle}(m,n)} f(u_{\sigma(1)},\ldots,u_{\sigma(m)})g(u_{\sigma(m+1)},\ldots,u_{\sigma(m+n)}),$$

where $\mathrm{shuffle}(m,n)$ consists of all (m,n)-"shuffles", that is, permutations, σ, of $\{1,\ldots m+n\}$, such that $\sigma(1)<\cdots<\sigma(m)$ and $\sigma(m+1)<\cdots<\sigma(m+n)$. We urge the reader to check this fact.

Another useful canonical isomorphim (of K-algebras) is

$$\mathrm{Sym}(E\oplus F)\cong\mathrm{Sym}(E)\otimes\mathrm{Sym}(F).$$

22.11 Exterior Tensor Powers

We now consider *alternating* (also called *skew-symmetric*) multilinear maps and *exterior tensor powers* (also called *alternating tensor powers*), denoted $\bigwedge^n(E)$. In many respect, alternating multilinear maps and exterior tensor powers can be treated much like symmetric tensor powers except that the sign, $\mathrm{sgn}(\sigma)$, needs to be inserted in front of the formulae valid for symmetric powers. Roughly speaking, we are now in the world of determinants rather than in the world of permanents. However, there are also some fundamental differences, one of which being that the exterior tensor power, $\bigwedge^n(E)$, is the trivial vector space, (0), when E is finite-dimensional and when $n>\dim(E)$. As in the case of symmetric tensor powers, since we already have the tensor algebra, $T(V)$, we can proceed rather quickly. But first, let us review some basic definitions and facts.

Definition 22.7 *Let $f\colon E^n\to F$ be a multilinear map. We say that f alternating iff $f(u_1,\ldots,u_n)=0$ whenever $u_i=u_{i+1}$, for some i with $1\leq i\leq n-1$, for all $u_i\in E$, that is, $f(u_1,\ldots,u_n)=0$ whenever two adjacent arguments are identical. We say that f is skew-symmetric (or anti-symmetric) iff*

$$f(u_{\sigma(1)},\ldots,u_{\sigma(n)})=\mathrm{sgn}(\sigma)f(u_1,\ldots,u_n),$$

for every permutation, $\sigma\in\mathfrak{S}_n$, and all $u_i\in E$.

For $n = 1$, we agree that every linear map, $f\colon E \to F$, is alternating. The vector space of all multilinear alternating maps, $f\colon E^n \to F$, is denoted $\mathrm{Alt}^n(E; F)$. Note that $\mathrm{Alt}^1(E; F) = \mathrm{Hom}(E, F)$. The following basic proposition shows the relationship between alternation and skew-symmetry.

Proposition 22.15 *Let $f\colon E^n \to F$ be a multilinear map. If f is alternating, then the following properties hold:*

(1) For all i, with $1 \leq i \leq n-1$,

$$f(\ldots, u_i, u_{i+1}, \ldots) = -f(\ldots, u_{i+1}, u_i, \ldots).$$

(2) For every permutation, $\sigma \in \mathfrak{S}_n$,

$$f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) = \mathrm{sgn}(\sigma) f(u_1, \ldots, u_n).$$

(3) For all i, j, with $1 \leq i < j \leq n$,

$$f(\ldots, u_i, \ldots u_j, \ldots) = 0 \quad \textit{whenever } u_i = u_j.$$

Moreover, if our field, K, has characteristic different from 2, then every skew-symmetric multilinear map is alternating.

Proof. (i) By multilinearity applied twice, we have

$$\begin{aligned} f(\ldots, u_i + u_{i+1}, u_i + u_{i+1}, \ldots) = f(\ldots, u_i, u_i, \ldots) + f(\ldots, u_i, u_{i+1}, \ldots) \\ + f(\ldots, u_{i+1}, u_i, \ldots) + f(\ldots, u_{i+1}, u_{i+1}, \ldots). \end{aligned}$$

Since f is alternating, we get

$$0 = f(\ldots, u_i, u_{i+1}, \ldots) + f(\ldots, u_{i+1}, u_i, \ldots),$$

that is, $f(\ldots, u_i, u_{i+1}, \ldots) = -f(\ldots, u_{i+1}, u_i, \ldots)$.

(ii) Clearly, the symmetric group, $\mathfrak{S}_n$, acts on $\mathrm{Alt}^n(E; F)$ on the left, *via*

$$\sigma \cdot f(u_1, \ldots, u_n) \ = f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}).$$

Consequently, as $\mathfrak{S}_n$ is generated by the transpositions (permutations that swap exactly two elements), since for a transposition, (ii) is simply (i), we deduce (ii) by induction on the number of transpositions in σ.

(iii) There is a permutation, σ, that sends u_i and u_j respectively to u_1 and u_2. As f is alternating,

$$f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) = 0.$$

However, by (ii),

$$f(u_1, \ldots, u_n) = \mathrm{sgn}(\sigma) f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) = 0.$$

Now, when f is skew-symmetric, if σ is the transposition swapping u_i and $u_{i+1} = u_i$, as $\mathrm{sgn}(\sigma) = -1$, we get

$$f(\ldots, u_i, u_i, \ldots) = -f(\ldots, u_i, u_i, \ldots),$$

so that

$$2f(\ldots, u_i, u_i, \ldots) = 0,$$

and in every characteristic except 2, we conclude that $f(\ldots, u_i, u_i, \ldots) = 0$, namely, f is alternating. $\square$

Proposition 22.15 shows that in every characteristic except 2, alternating and skew-symmetric multilinear maps are identical. Using Proposition 22.15 we easily deduce the following crucial fact:

Proposition 22.16 *Let $f\colon E^n \to F$ be an alternating multilinear map. For any families of vectors, $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$, with $u_i, v_i \in E$, if*

$$v_j = \sum_{i=1}^{n} a_{ij} u_i, \qquad 1 \leq j \leq n,$$

then

$$f(v_1, \ldots, v_n) = \left(\sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\, a_{\sigma(1),1} \cdots a_{\sigma(n),n} \right) f(u_1, \ldots, u_n) = \det(A) f(u_1, \ldots, u_n),$$

where A is the $n \times n$ matrix, $A = (a_{ij})$.

Proof. Use property (ii) of Proposition 22.15. $\square$.

We are now ready to define and construct exterior tensor powers.

Definition 22.8 An *n-th exterior tensor power* of a vector space, E, where $n \geq 1$, is a vector space, A, together with an alternating multilinear map, $\varphi\colon E^n \to A$, such that, for every vector space, F, and for every alternating multilinear map, $f\colon E^n \to F$, there is a unique linear map, $f_\wedge\colon A \to F$, with

$$f(u_1, \ldots, u_n) = f_\wedge(\varphi(u_1, \ldots, u_n)),$$

for all $u_1, \ldots, u_n \in E$, or for short

$$f = f_\wedge \circ \varphi.$$

Equivalently, there is a unique linear map $f_\wedge$ such that the following diagram commutes:

$$\begin{tikzcd} E^n \arrow[r, "\varphi"] \arrow[dr, "f"'] & A \arrow[d, "f_\wedge"] \\ & F \end{tikzcd}$$

First, we show that any two n-th exterior tensor powers (A_1, φ_1) and (A_2, φ_2) for E, are isomorphic.

Proposition 22.17 *Given any two n-th exterior tensor powers (A_1, φ_1) and (A_2, φ_2) for E, there is an isomorphism $h\colon A_1 \to A_2$ such that*

$$\varphi_2 = h \circ \varphi_1.$$

Proof. Replace tensor product by n exterior tensor power in the proof of Proposition 22.4. $\square$

We now give a construction that produces an n-th exterior tensor power of a vector space E.

Theorem 22.18 *Given a vector space E, an n-th exterior tensor power $(\bigwedge^n(E), \varphi)$ for E can be constructed ($n \geq 1$). Furthermore, denoting $\varphi(u_1, \ldots, u_n)$ as $u_1 \wedge \cdots \wedge u_n$, the exterior tensor power $\bigwedge^n(E)$ is generated by the vectors $u_1 \wedge \cdots \wedge u_n$, where $u_1, \ldots, u_n \in E$, and for every alternating multilinear map $f\colon E^n \to F$, the unique linear map $f_\wedge\colon \bigwedge^n(E) \to F$ such that $f = f_\wedge \circ \varphi$, is defined by*

$$f_\wedge(u_1 \wedge \cdots \wedge u_n) = f(u_1, \ldots, u_n),$$

on the generators $u_1 \wedge \cdots \wedge u_n$ of $\bigwedge^n(E)$.

Proof sketch. We can give a quick proof using the tensor algebra, $T(E)$. let $\mathfrak{I}_a$ be the two-sided ideal of $T(E)$ generated by all tensors of the form $u \otimes u \in E^{\otimes 2}$. Then, let

$$\bigwedge^n(E) = E^{\otimes n}/(\mathfrak{I}_a \cap E^{\otimes n})$$

and let π be the projection, $\pi\colon E^{\otimes n} \to \bigwedge^n(E)$. If we let $u_1 \wedge \cdots \wedge u_n = \pi(u_1 \otimes \cdots \otimes u_n)$, it is easy to check that $(\bigwedge^n(E), \wedge)$ satisfies the conditions of Theorem 22.18. $\square$

Remark: We can also define

$$\bigwedge(E) = T(E)/\mathfrak{I}_a = \bigoplus_{n\geq 0} \bigwedge^n(E),$$

the *exterior algebra* of E. This is the skew-symmetric counterpart of $\mathrm{Sym}(E)$ and we will study it a little later.

For simplicity of notation, we may write $\bigwedge^n E$ for $\bigwedge^n(E)$. We also abbreviate "exterior tensor power" as "exterior power". Clearly, $\bigwedge^1(E) \cong E$ and it is convenient to set $\bigwedge^0(E) = K$.

The fact that the map $\varphi\colon E^n \to \bigwedge^n(E)$ is alternating and multinear, can also be expressed as follows:

$$\begin{aligned} u_1 \wedge \cdots \wedge (u_i + v_i) \wedge \cdots \wedge u_n &= (u_1 \wedge \cdots \wedge u_i \wedge \cdots \wedge u_n) \\ &\quad + (u_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge u_n), \\ u_1 \wedge \cdots \wedge (\lambda u_i) \wedge \cdots \wedge u_n &= \lambda(u_1 \wedge \cdots \wedge u_i \wedge \cdots \wedge u_n), \\ u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(n)} &= \mathrm{sgn}(\sigma)\, u_1 \wedge \cdots \wedge u_n, \end{aligned}$$

for all $\sigma \in \mathfrak{S}_n$.

Theorem 22.18 yields a canonical isomorphism

$$\mathrm{Hom}(\bigwedge^n(E), F) \cong \mathrm{Alt}^n(E; F),$$

between the vector space of linear maps $\mathrm{Hom}(\bigwedge^n(E), F)$, and the vector space of alternating multilinear maps $\mathrm{Alt}^n(E; F)$, via the linear map $- \circ \varphi$ defined by

$$h \mapsto h \circ \varphi,$$

where $h \in \mathrm{Hom}(\bigwedge^n(E), F)$. In particular, when $F = K$, we get a canonical isomorphism

$$\left(\bigwedge^n(E)\right)^* \cong \mathrm{Alt}^n(E; K).$$

Tensors $\alpha \in \bigwedge^n(E)$ are called *alternating n-tensors* or *alternating tensors of degree n* and we write $\deg(\alpha) = n$. Tensors of the form $u_1 \wedge \cdots \wedge u_n$, where $u_i \in E$, are called *simple (or decomposable) alternating n-tensors*. Those alternating n-tensors that are not simple are often called *compound alternating n-tensors*. Simple tensors $u_1 \wedge \cdots \wedge u_n \in \bigwedge^n(E)$ are also called *n-vectors* and tensors in $\bigwedge^n(E^*)$ are often called *(alternating) n-forms*.

Given two linear maps $f\colon E \to E'$ and $g\colon E \to E'$, we can define $h\colon E \times E \to \bigwedge^2(E')$ by

$$h(u, v) = f(u) \wedge g(v).$$

It is immediately verified that h is alternating bilinear, and thus, it induces a unique linear map

$$f \wedge g\colon \bigwedge^2(E) \to \bigwedge^2(E'),$$

such that

$$(f \wedge g)(u \wedge v) = f(u) \wedge g(u).$$

If we also have linear maps $f'\colon E' \to E''$ and $g'\colon E' \to E''$, we can easily verify that

$$(f' \circ f) \wedge (g' \circ g) = (f' \wedge g') \circ (f \wedge g).$$

The generalization to the alternating product $f_1 \wedge \cdots \wedge f_n$ of $n \geq 3$ linear maps $f_i\colon E \to E'$ is immediate, and left to the reader.

22.12 Bases of Exterior Powers

Let E be any vector space. For any basis, $(u_i)_{i\in\Sigma}$, for E, we assume that some total ordering, $\leq$, on Σ, has been chosen. Call the pair $((u_i)_{i\in\Sigma}, \leq)$ an *ordered basis*. Then, for any nonempty finite subset, $I \subseteq \Sigma$, let

$$u_I = u_{i_1} \wedge \cdots \wedge u_{i_m},$$

where $I = \{i_1, \ldots, i_m\}$, with $i_1 < \cdots < i_m$.

Since $\bigwedge^n(E)$ is generated by the tensors of the form $v_1 \wedge \cdots \wedge v_n$, with $v_i \in E$, in view of skew-symmetry, it is clear that the tensors u_I, with $|I| = n$, generate $\bigwedge^n(E)$. Actually, they form a basis.

Proposition 22.19 *Given any vector space, E, if E has finite dimension, $d = \dim(E)$, then for all $n > d$, the exterior power $\bigwedge^n(E)$ is trivial, that is $\bigwedge^n(E) = (0)$. Otherwise, for every ordered basis, $((u_i)_{i\in\Sigma}, \leq)$, the family, (u_I), is basis of $\bigwedge^n(E)$, where I ranges over finite nonempty subsets of Σ of size $|I| = n$.*

Proof. First, assume that E has finite dimension, $d = \dim(E)$ and that $n > d$. We know that $\bigwedge^n(E)$ is generated by the tensors of the form $v_1 \wedge \cdots \wedge v_n$, with $v_i \in E$. If $u_1, \ldots, u_d$ is a basis of E, as every v_i is a linear combination of the u_j, when we expand $v_1 \wedge \cdots \wedge v_n$ using multilinearity, we get a linear combination of the form

$$v_1 \wedge \cdots \wedge v_n = \sum_{(j_1,\ldots,j_n)} \lambda_{(j_1,\ldots,j_n)}\, u_{j_1} \wedge \cdots \wedge u_{j_n},$$

where each $(j_1, \ldots, j_n)$ is some sequence of integers $j_k \in \{1, \ldots, d\}$. As $n > d$, each sequence $(j_1, \ldots, j_n)$ must contain two identical elements. By alternation, $u_{j_1} \wedge \cdots \wedge u_{j_n} = 0$ and so, $v_1 \wedge \cdots \wedge v_n = 0$. It follows that $\bigwedge^n(E) = (0)$.

Now, assume that either $\dim(E) = d$ and that $n \leq d$ or that E is infinite dimensional. The argument below shows that the u_I are nonzero and linearly independent. As usual, let $u_i^* \in E^*$ be the linear form given by

$$u_i^*(u_j) = \delta_{ij}.$$

For any nonempty subset, $I = \{i_1, \ldots, i_n\} \subseteq \Sigma$, with $i_1 < \cdots < i_n$, let l_I be the map given by

$$l_I(v_1, \ldots, v_n) = \det(u_{i_j}^*(v_k)),$$

for all $v_k \in E$. As l_I is alternating multilinear, it induces a linear map, $L_I \colon \bigwedge^n(E) \to K$. Observe that for any nonempty finite subset, $J \subseteq \Sigma$, with $|J| = n$, we have

$$L_I(u_J) = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J. \end{cases}$$

Note that when $\dim(E) = d$ and $n \leq d$, the forms $u^*_{i_1}, \ldots, u^*_{i_n}$ are all distinct so, the above does hold. Since $L_I(u_I) = 1$, we conclude that $u_I \neq 0$. Now, if we have a linear combination,

$$\sum_I \lambda_I u_I = 0,$$

where the above sum is finite and involves nonempty finite subset, $I \subseteq \Sigma$, with $|I| = n$, for every such I, when we apply L_I we get

$$\lambda_I = 0,$$

proving linear independence. $\square$

As a corollary, if E is finite dimensional, say $\dim(E) = d$ and if $1 \leq n \leq d$, then we have

$$\dim(\bigwedge^n(E)) = \binom{n}{d}$$

and if $n > d$, then $\dim(\bigwedge^n(E)) = 0$.

Remark: When $n = 0$, if we set $u_\emptyset = 1$, then $(u_\emptyset) = (1)$ is a basis of $\bigwedge^0(V) = K$.

It follows from Proposition 22.19 that the family, $(u_I)_I$, where $I \subseteq \Sigma$ ranges over finite subsets of Σ is a basis of $\bigwedge(V) = \bigoplus_{n\geq 0} \bigwedge^n(V)$.

As a corollary of Proposition 22.19 we obtain the following useful criterion for linear independence:

Proposition 22.20 *For any vector space, E, the vectors, $u_1, \ldots, u_n \in E$, are linearly independent iff $u_1 \wedge \cdots \wedge u_n \neq 0$.*

Proof. If $u_1 \wedge \cdots \wedge u_n \neq 0$, then $u_1, \ldots, u_n$ must be linearly independent. Otherwise, some u_i would be a linear combination of the other u_j's (with $j \neq i$) and then, as in the proof of Proposition 22.19, $u_1 \wedge \cdots \wedge u_n$ would be a linear combination of wedges in which two vectors are identical and thus, zero.

Conversely, assume that $u_1, \ldots, u_n$ are linearly independent. Then, we have the linear forms, $u^*_i \in E^*$, such that

$$u^*_i(u_j) = \delta_{i,j} \qquad 1 \leq i, j \leq n.$$

As in the proof of Proposition 22.19, we have a linear map, $L_{u_1,\ldots,u_n} \colon \bigwedge^n(E) \to K$, given by

$$L_{u_1,\ldots,u_n}(v_1 \wedge \cdots \wedge v_n) = \det(u^*_j(v_i)),$$

for all $v_1 \wedge \cdots \wedge v_n \in \bigwedge^n(E)$. As,

$$L_{u_1,\ldots,u_n}(u_1 \wedge \cdots \wedge u_n) = 1,$$

we conclude that $u_1 \wedge \cdots \wedge u_n \neq 0$. $\square$

Proposition 22.20 shows that, geometrically, every nonzero wedge, $u_1 \wedge \cdots \wedge u_n$, corresponds to some oriented version of an n-dimensional subspace of E.

22.13 Some Useful Isomorphisms for Exterior Powers

We can show the following property of the exterior tensor product, using the proof technique of Proposition 22.7:

$$\bigwedge^n(E \oplus F) \cong \bigoplus_{k=0}^{n} \bigwedge^k(E) \otimes \bigwedge^{n-k}(F).$$

22.14 Duality for Exterior Powers

In this section, all vector spaces are assumed to have finite dimension. We define a nondegenerate pairing, $\bigwedge^n(E^*) \times \bigwedge^n(E) \longrightarrow K$, as follows: Consider the multilinear map,

$$(E^*)^n \times E^n \longrightarrow K,$$

given by

$$(v_1^*, \ldots, v_n^*, u_1, \ldots, u_n) \mapsto \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\, v_{\sigma(1)}^*(u_1) \cdots v_{\sigma(n)}^*(u_n) = \det(v_j^*(u_i)).$$

It is easily checked that this expression is alternating w.r.t. the u_i's and also w.r.t. the v_j^*. For any fixed $(v_1^*, \ldots, v_n^*) \in (E^*)^n$, we get an alternating multinear map,

$$l_{v_1^*, \ldots, v_n^*} \colon (u_1, \ldots, u_n) \mapsto \det(v_j^*(u_i)),$$

from E^n to K. By the argument used in the symmetric case, we get a bilinear map,

$$\bigwedge^n(E^*) \times \bigwedge^n(E) \longrightarrow K.$$

Now, this pairing in nondegenerate. This can be done using bases and we leave it as an exercise to the reader. Therefore, we get a canonical isomorphism,

$$(\bigwedge^n(E))^* \cong \bigwedge^n(E^*).$$

Since we also have a canonical isomorphism

$$(\bigwedge^n(E))^* \cong \mathrm{Alt}^n(E; K),$$

we get a canonical isomorphism

$$\bigwedge^n(E^*) \cong \mathrm{Alt}^n(E; K)$$

which allows us to interpret alternating tensors over E^* as alternating multilinear maps.

The isomorphism, $\mu\colon \bigwedge^n(E^*) \cong \mathrm{Alt}^n(E;K)$, discussed above can be described explicity as the linear extension of the map given by

$$\mu(v_1^* \wedge \cdots \wedge v_n^*)(u_1, \ldots, u_n) = \det(v_j^*(u_i)).$$

Remark: Variants of our isomorphism, μ, are found in the literature. For example, there is a version, μ', where

$$\mu' = \frac{1}{n!}\mu,$$

with the factor $\frac{1}{n!}$ added in front of the determinant. Each version has its its own merits and inconvenients. Morita [114] uses μ' because it is more convenient than μ when dealing with characteristic classes. On the other hand, when using μ', some extra factor is needed in defining the wedge operation of alternating multilinear forms (see Section 22.15) and for exterior differentiation. The version μ is the one adopted by Warner [147], Knapp [89], Fulton and Harris [57] and Cartan [29, 30].

If $f\colon E \to F$ is any linear map, by transposition we get a linear map, $f^\top\colon F^* \to E^*$, given by

$$f^\top(v^*) = v^* \circ f, \qquad v^* \in F^*.$$

Consequently, we have

$$f^\top(v^*)(u) = v^*(f(u)), \qquad \text{for all}\, u \in E \,\text{and all}\, v^* \in F^*.$$

For any $p \geq 1$, the map,

$$(u_1, \ldots, u_p) \mapsto f(u_1) \wedge \cdots \wedge f(u_p),$$

from E^n to $\bigwedge^p F$ is multilinear alternating, so it induces a linear map, $\bigwedge^p f\colon \bigwedge^p E \to \bigwedge^p F$, defined on generators by

$$\Big(\bigwedge^p f\Big)(u_1 \wedge \cdots \wedge u_p) = f(u_1) \wedge \cdots \wedge f(u_p).$$

Combining $\bigwedge^p$ and duality, we get a linear map, $\bigwedge^p f^\top\colon \bigwedge^p F^* \to \bigwedge^p E^*$, defined on generators by

$$\Big(\bigwedge^p f^\top\Big)(v_1^* \wedge \cdots \wedge v_p^*) = f^\top(v_1^*) \wedge \cdots \wedge f^\top(v_p^*).$$

Proposition 22.21 *If $f\colon E \to F$ is any linear map between two finite-dimensional vector spaces, E and F, then*

$$\mu\Big(\Big(\bigwedge^p f^\top\Big)(\omega)\Big)(u_1, \ldots, u_p) = \mu(\omega)(f(u_1), \ldots, f(u_p)), \qquad \omega \in \bigwedge^p F^*,\ u_1, \ldots, u_p \in E.$$

Proof. It is enough to prove the formula on generators. By definition of μ, we have

$$\begin{aligned}
\mu\Big(\Big(\bigwedge^p f^\top\Big)(v_1^*\wedge\cdots\wedge v_p^*)\Big)(u_1,\ldots,u_p) &= \mu(f^\top(v_1^*)\wedge\cdots\wedge f^\top(v_p^*))(u_1,\ldots,u_p)\\
&= \det(f^\top(v_j^*)(u_i))\\
&= \det(v_j^*(f(u_i)))\\
&= \mu(v_1^*\wedge\cdots\wedge v_p^*)(f(u_1),\ldots,f(u_p)),
\end{aligned}$$

as claimed. □

The map $\bigwedge^p f^\top$ is often denoted f^*, although this is an ambiguous notation since p is dropped. Proposition 22.21 gives us the behavior of f^* under the identification of $\bigwedge^p E^*$ and $\mathrm{Alt}^p(E;K)$ *via* the isomorphism μ.

As in the case of symmetric powers, the map from E^n to $\bigwedge^n(E)$ given by $(u_1,\ldots,u_n)\mapsto u_1\wedge\cdots\wedge u_n$ yields a surjection, $\pi\colon E^{\otimes n}\to\bigwedge^n(E)$. Now, this map has some section so there is some injection, $\iota\colon \bigwedge^n(E)\to E^{\otimes n}$, with $\pi\circ\iota=\mathrm{id}$. If our field, K, has characteristic 0, then there is a special section having a natural definition involving an antisymmetrization process.

Recall that we have a left action of the symmetric group, $\mathfrak{S}_n$, on $E^{\otimes n}$. The tensors, $z\in E^{\otimes n}$, such that

$$\sigma\cdot z=\mathrm{sgn}(\sigma)\,z,\quad \text{for all}\quad \sigma\in\mathfrak{S}_n$$

are called *antisymmetrized* tensors. We define the map, $\iota\colon E^n\to E^{\otimes n}$, by

$$\iota(u_1,\ldots,u_n)=\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\mathrm{sgn}(\sigma)\,u_{\sigma(1)}\otimes\cdots\otimes u_{\sigma(n)}.$$

As the right hand side is clearly an alternating map, we get a linear map, $\iota\colon \bigwedge^n(E)\to E^{\otimes n}$. Clearly, $\iota(\bigwedge^n(E))$ is the set of antisymmetrized tensors in $E^{\otimes n}$. If we consider the map, $A=\iota\circ\pi\colon E^{\otimes n}\longrightarrow E^{\otimes n}$, it is easy to check that $A\circ A=A$. Therefore, A is a projection and by linear algebra, we know that

$$E^{\otimes n}=A(E^{\otimes n})\oplus\mathrm{Ker}\,A=\iota(\bigwedge^n(A))\oplus\mathrm{Ker}\,A.$$

It turns out that $\mathrm{Ker}\,A=E^{\otimes n}\cap\mathfrak{I}_a=\mathrm{Ker}\ \pi$, where $\mathfrak{I}_a$ is the two-sided ideal of $T(E)$ generated by all tensors of the form $u\otimes u\in E^{\otimes 2}$ (for example, see Knapp [89], Appendix A). Therefore, ι is injective,

$$E^{\otimes n}=\iota(\bigwedge^n(E))\oplus E^{\otimes n}\cap\mathfrak{I}=\iota(\bigwedge^n(E))\oplus\mathrm{Ker}\,\pi,$$

and the exterior tensor power, $\bigwedge^n(E)$, is naturally embedded into $E^{\otimes n}$.

22.15 Exterior Algebras

As in the case of symmetric tensors, we can pack together all the exterior powers, $\bigwedge^n(V)$, into an algebra,

$$\bigwedge(V) = \bigoplus_{m\geq 0} \bigwedge^m(V),$$

called the *exterior algebra (or Grassmann algebra) of* V. We mimic the procedure used for symmetric powers. If $\mathfrak{I}_a$ is the two-sided ideal generated by all tensors of the form $u \otimes u \in V^{\otimes 2}$, we set

$$\bigwedge^{\bullet}(V) = T(V)/\mathfrak{I}_a.$$

Then, $\bigwedge^{\bullet}(V)$ automatically inherits a multiplication operation, called *wedge product*, and since $T(V)$ is graded, that is,

$$T(V) = \bigoplus_{m\geq 0} V^{\otimes m},$$

we have

$$\bigwedge^{\bullet}(V) = \bigoplus_{m\geq 0} V^{\otimes m}/(\mathfrak{I}_a \cap V^{\otimes m}).$$

However, it is easy to check that

$$\bigwedge^m(V) \cong V^{\otimes m}/(\mathfrak{I}_a \cap V^{\otimes m}),$$

so

$$\bigwedge^{\bullet}(V) \cong \bigwedge(V).$$

When V has finite dimension, d, we actually have a finite coproduct

$$\bigwedge(V) = \bigoplus_{m=0}^{d} \bigwedge^m(V),$$

and since each $\bigwedge^m(V)$ has dimension, $\binom{d}{m}$, we deduce that

$$\dim(\bigwedge(V)) = 2^d = 2^{\dim(V)}.$$

The multiplication, $\wedge\colon \bigwedge^m(V) \times \bigwedge^n(V) \to \bigwedge^{m+n}(V)$, is skew-symmetric in the following precise sense:

Proposition 22.22 *For all $\alpha \in \bigwedge^m(V)$ and all $\beta \in \bigwedge^n(V)$, we have*

$$\beta \wedge \alpha = (-1)^{mn} \alpha \wedge \beta.$$

Proof. Since $v \wedge u = -u \wedge v$ for all $u, v \in V$, Proposition 22.22 follows by induction. $\square$

Since $\alpha \wedge \alpha = 0$ for every *simple* tensor, $\alpha = u_1 \wedge \cdots \wedge u_n$, it seems natural to infer that $\alpha \wedge \alpha = 0$ for *every* tensor $\alpha \in \bigwedge(V)$. If we consider the case where $\dim(V) \leq 3$, we can indeed prove the above assertion. However, if $\dim(V) \geq 4$, the above fact is generally false! For example, when $\dim(V) = 4$, if u_1, u_2, u_3, u_4 are a basis for V, for $\alpha = u_1 \wedge u_2 + u_3 \wedge u_4$, we check that

$$\alpha \wedge \alpha = 2u_1 \wedge u_2 \wedge u_3 \wedge u_4,$$

which is nonzero.

The above discussion suggests that it might be useful to know when an alternating tensor is simple, that is, decomposable. It can be shown that for tensors, $\alpha \in \bigwedge^2(V)$, $\alpha \wedge \alpha = 0$ iff α is simple. A general criterion for decomposability can be given in terms of some operations known as *left hook* and *right hook* (also called *interior products*), see Section 22.17.

It is easy to see that $\bigwedge(V)$ satisfies the following universal mapping property:

Proposition 22.23 *Given any K-algebra, A, for any linear map, $f\colon V \to A$, if $(f(v))^2 = 0$ for all $v \in V$, then there is a unique K-algebra homomorphism, $\overline{f}\colon \bigwedge(V) \to A$, so that*

$$f = \overline{f} \circ i,$$

as in the diagram below:

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "f"'] & \bigwedge(V) \arrow[d, "\overline{f}"] \\ & A \end{tikzcd}$$

When E is finite-dimensional, recall the isomorphism, $\mu\colon \bigwedge^n(E^*) \longrightarrow \mathrm{Alt}^n(E; K)$, defined as the linear extension of the map given by

$$\mu(v_1^* \wedge \cdots \wedge v_n^*)(u_1, \ldots, u_n) = \det(u_j^*(u_i)).$$

Now, we have also a multiplication operation, $\bigwedge^m(E^*) \times \bigwedge^n(E^*) \longrightarrow \bigwedge^{m+n}(E^*)$. The following question then arises:

Can we define a multiplication, $\mathrm{Alt}^m(E; K) \times \mathrm{Alt}^n(E; K) \longrightarrow \mathrm{Alt}^{m+n}(E; K)$, directly on alternating multilinear forms, so that the following diagram commutes:

$$\begin{tikzcd} \bigwedge^m(E^*) \times \bigwedge^n(E^*) \arrow[r, "\wedge"] \arrow[d, "\mu \times \mu"] & \bigwedge^{m+n}(E^*) \arrow[d, "\mu"] \\ \mathrm{Alt}^m(E; K) \times \mathrm{Alt}^n(E; K) \arrow[r, "\wedge"] & \mathrm{Alt}^{m+n}(E; K). \end{tikzcd}$$

As in the symmetric case, the answer is *yes*! The solution is to define this multiplication such that, for $f \in \mathrm{Alt}^m(E; K)$ and $g \in \mathrm{Alt}^n(E; K)$,

$$(f \wedge g)(u_1, \ldots, u_{m+n}) = \sum_{\sigma \in \mathrm{shuffle}(m,n)} \mathrm{sgn}(\sigma)\, f(u_{\sigma(1)}, \ldots, u_{\sigma(m)}) g(u_{\sigma(m+1)}, \ldots, u_{\sigma(m+n)}),$$

where $\mathrm{shuffle}(m,n)$ consists of all (m,n)-"shuffles", that is, permutations, σ, of $\{1,\ldots m+n\}$, such that $\sigma(1)<\cdots<\sigma(m)$ and $\sigma(m+1)<\cdots<\sigma(m+n)$. For example, when $m=n=1$, we have

$$(f\wedge g)(u,v)=f(u)g(v)-g(u)f(v).$$

When $m=1$ and $n\geq 2$, check that

$$(f\wedge g)(u_1,\ldots,u_{m+1})=\sum_{i=1}^{m+1}(-1)^{i-1}f(u_i)g(u_1,\ldots,\widehat{u_i},\ldots,u_{m+1}),$$

where the hat over the argument u_i means that it should be omitted.

As a result of all this, the coproduct

$$\mathrm{Alt}(E)=\bigoplus_{n\geq 0}\mathrm{Alt}^n(E;K)$$

is an algebra under the above multiplication and this algebra is isomorphic to $\bigwedge(E^*)$. For the record, we state

Proposition 22.24 *When E is finite dimensional, the maps, $\mu\colon \bigwedge^n(E^*)\longrightarrow \mathrm{Alt}^n(E;K)$, induced by the linear extensions of the maps given by*

$$\mu(v_1^*\wedge\cdots\wedge v_n^*)(u_1,\ldots,u_n)=\det(u_j^*(u_i))$$

yield a canonical isomorphism of algebras, $\mu\colon \bigwedge(E^*)\longrightarrow \mathrm{Alt}(E)$, where the multiplication in $\mathrm{Alt}(E)$ is defined by the maps, $\wedge\colon \mathrm{Alt}^m(E;K)\times\mathrm{Alt}^n(E;K)\longrightarrow\mathrm{Alt}^{m+n}(E;K)$, with

$$(f\wedge g)(u_1,\ldots,u_{m+n})=\sum_{\sigma\in\mathrm{shuffle}(m,n)}\mathrm{sgn}(\sigma)\,f(u_{\sigma(1)},\ldots,u_{\sigma(m)})g(u_{\sigma(m+1)},\ldots,u_{\sigma(m+n)}),$$

where $\mathrm{shuffle}(m,n)$ consists of all (m,n)-"shuffles", that is, permutations, σ, of $\{1,\ldots m+n\}$, such that $\sigma(1)<\cdots<\sigma(m)$ and $\sigma(m+1)<\cdots<\sigma(m+n)$.

Remark: The algebra, $\bigwedge(E)$ is a graded algebra. Given two graded algebras, E and F, we can make a new tensor product, $E\,\widehat{\otimes}\,F$, where $E\,\widehat{\otimes}\,F$ is equal to $E\otimes F$ as a vector space, but with a skew-commutative multiplication given by

$$(a\otimes b)\wedge(c\otimes d)=(-1)^{\deg(b)\deg(c)}(ac)\otimes(bd),$$

where $a\in E^m, b\in F^p$, $c\in E^n, d\in F^q$. Then, it can be shown that

$$\bigwedge(E\oplus F)\cong\bigwedge(E)\,\widehat{\otimes}\,\bigwedge(F).$$

22.16 The Hodge $*$-Operator

In order to define a generalization of the Laplacian that will apply to differential forms on a Riemannian manifold, we need to define isomorphisms,

$$\bigwedge^k V \longrightarrow \bigwedge^{n-k} V,$$

for any Euclidean vector space, V, of dimension n and any k, with $0 \leq k \leq n$. If $\langle -, - \rangle$ denotes the inner product on V, we define an inner product on $\bigwedge^k V$, also denoted $\langle -, - \rangle$, by setting

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle = \det(\langle u_i, v_j \rangle),$$

for all $u_i, v_i \in V$ and extending $\langle -, - \rangle$ by bilinearity.

It is easy to show that if $(e_1, \ldots, e_n)$ is an orthonormal basis of V, then the basis of $\bigwedge^k V$ consisting of the e_I (where $I = \{i_1, \ldots, i_k\}$, with $1 \leq i_1 < \cdots < i_k \leq n$) is an orthonormal basis of $\bigwedge^k V$. Since the inner product on V induces an inner product on V^* (recall that $\langle \omega_1, \omega_2 \rangle = \langle \omega_1^\sharp, \omega_2^\sharp \rangle$, for all $\omega_1, \omega_2 \in V^*$), we also get an inner product on $\bigwedge^k V^*$.

Recall that an *orientation* of a vector space, V, of dimension n is given by the choice of some basis, $(e_1, \ldots, e_n)$. We say that a basis, $(u_1, \ldots, u_n)$, of V is *positively oriented* iff $\det(u_1, \ldots, u_n) > 0$ (where $\det(u_1, \ldots, u_n)$ denotes the determinant of the matrix whose jth column consists of the coordinates of u_j over the basis $(e_1, \ldots, e_n)$), otherwise it is *negatively oriented*. An *oriented vector space* is a vector space, V, together with an orientation of V. If V is oriented by the basis $(e_1, \ldots, e_n)$, then V^* is oriented by the dual basis, $(e_1^*, \ldots, e_n^*)$.

If V is an oriented vector space of dimension n, then we can define a linear map,

$$*\colon \bigwedge^k V \to \bigwedge^{n-k} V,$$

called the *Hodge $*$-operator*, as follows: For any choice of a positively oriented orthonormal basis, $(e_1, \ldots, e_n)$, of V, set

$$*(e_1 \wedge \cdots \wedge e_k) = e_{k+1} \wedge \cdots \wedge e_n.$$

In particular, for $k = 0$ and $k = n$, we have

$$\begin{aligned} *(1) &= e_1 \wedge \cdots \wedge e_n \\ *(e_1 \wedge \cdots \wedge e_n) &= 1. \end{aligned}$$

It is easy to see that the definition of $*$ does not depend on the choice of positively oriented orthonormal basis.

The Hodge $*$-operators, $*\colon \bigwedge^k V \to \bigwedge^{n-k} V$, induces a linear bijection, $*\colon \bigwedge(V) \to \bigwedge(V)$. We also have Hodge $*$-operators, $*\colon \bigwedge^k V^* \to \bigwedge^{n-k} V^*$.

The following proposition is easy to show:

Proposition 22.25 *If V is any oriented vector space of dimension n, for every k, with $0 \leq k \leq n$, we have*

(i) $** = (-\mathrm{id})^{k(n-k)}$.

(ii) $\langle x, y\rangle = *(x \wedge *y) = *(y \wedge *x)$, *for all* $x, y \in \bigwedge^k V$.

If $(e_1, \ldots, e_n)$ is an orthonormal basis of V and $(v_1, \ldots, v_n)$ is any other basis of V, it is easy to see that

$$v_1 \wedge \cdots \wedge v_n = \sqrt{\det(\langle v_i, v_j\rangle)}\, e_1 \wedge \cdots \wedge e_n,$$

from which it follows that

$$*(1) = \frac{1}{\sqrt{\det(\langle v_i, v_j\rangle)}}\, v_1 \wedge \cdots \wedge v_n$$

(see Jost [83], Chapter 2, Lemma 2.1.3).

22.17 Testing Decomposability; Left and Right Hooks

In this section, all vector spaces are assumed to have finite dimension. Say $\dim(E) = n$. Using our nonsingular pairing,

$$\langle -, -\rangle \colon \bigwedge^p E^* \times \bigwedge^p E \longrightarrow K \qquad (1 \leq p \leq n),$$

defined on generators by

$$\langle u_1^* \wedge \cdots \wedge u_p^*, v_1 \wedge \cdots \wedge u_p\rangle = \det(u_i^*(v_j)),$$

we define various contraction operations,

$$\lrcorner \colon \bigwedge^p E \times \bigwedge^{p+q} E^* \longrightarrow \bigwedge^q E^* \qquad \text{(left hook)}$$

and

$$\llcorner \colon \bigwedge^{p+q} E^* \times \bigwedge^p E \longrightarrow \bigwedge^q E^* \qquad \text{(right hook)},$$

as well as the versions obtained by replacing E by E^* and E^{**} by E. We begin with the *left interior product or left hook*, $\lrcorner$.

Let $u \in \bigwedge^p E$. For any q such that $p + q \leq n$, multiplication on the right by u is a linear map

$$\wedge_R(u) \colon \bigwedge^q E \longrightarrow \bigwedge^{p+q} E,$$

given by

$$v \mapsto v \wedge u$$

where $v \in \bigwedge^q E$. The transpose of $\wedge_R(u)$ yields a linear map,

$$(\wedge_R(u))^t \colon (\bigwedge^{p+q} E)^* \longrightarrow (\bigwedge^{q} E)^*,$$

which, using the isomorphisms $(\bigwedge^{p+q} E)^* \cong \bigwedge^{p+q} E^*$ and $(\bigwedge^q E)^* \cong \bigwedge^q E^*$ can be viewed as a map

$$(\wedge_R(u))^t \colon \bigwedge^{p+q} E^* \longrightarrow \bigwedge^{q} E^*,$$

given by

$$z^* \mapsto z^* \circ \wedge_R(u),$$

where $z^* \in \bigwedge^{p+q} E^*$.

We denote $z^* \circ \wedge_R(u)$ by

$$u \lrcorner\, z^*.$$

In terms of our pairing, the q-vector $u \lrcorner\, z^*$ is uniquely defined by

$$\langle u \lrcorner\, z^*, v\rangle = \langle z^*, v \wedge u\rangle, \quad \text{for all } u \in \textstyle\bigwedge^p E,\ v \in \bigwedge^q E \text{ and } z^* \in \bigwedge^{p+q} E^*.$$

It is immediately verified that

$$(u \wedge v) \lrcorner\, z^* = u \lrcorner\, (v \lrcorner\, z^*),$$

so $\lrcorner$ defines a left action

$$\lrcorner \colon \bigwedge^{p} E \times \bigwedge^{p+q} E^* \longrightarrow \bigwedge^{q} E^*.$$

By interchanging E and E^* and using the isomorphism,

$$(\bigwedge^{k} F)^* \cong \bigwedge^{k} F^*,$$

we can also define a left action

$$\lrcorner \colon \bigwedge^{p} E^* \times \bigwedge^{p+q} E \longrightarrow \bigwedge^{q} E.$$

In terms of our pairing, $u^* \lrcorner\, z$ is uniquely defined by

$$\langle v^*, u^* \lrcorner\, z\rangle = \langle v^* \wedge u^*, z\rangle, \quad \text{for all } u^* \in \textstyle\bigwedge^p E^*,\ v^* \in \bigwedge^q E^* \text{ and } z \in \bigwedge^{p+q} E.$$

In order to proceed any further, we need some combinatorial properties of the basis of $\bigwedge^p E$ constructed from a basis, $(e_1, \ldots, e_n)$, of E. Recall that for any (nonempty) subset, $I \subseteq \{1, \ldots, n\}$, we let

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_p},$$

where $I = \{i_1, \ldots, i_p\}$ with $i_1 < \cdots < i_p$. We also let $e_\emptyset = 1$.

Given any two subsets $H, L \subseteq \{1, \ldots, n\}$, let

$$\rho_{H,L} = \begin{cases} 0 & \text{if } H \cap L \neq \emptyset, \\ (-1)^\nu & \text{if } H \cap L = \emptyset, \end{cases}$$

where

$$\nu = |\{(h, l) \mid (h, l) \in H \times L,\ h > l\}|.$$

Proposition 22.26 *For any basis, $(e_1, \ldots, e_n)$, of E the following properties hold:*

(1) If $H \cap L = \emptyset$, $|H| = h$, and $|L| = l$, then

$$\rho_{H,L}\rho_{L,H} = (-1)^{hl}.$$

(2) For $H, L \subseteq \{1, \ldots, m\}$, we have

$$e_H \wedge e_L = \rho_{H,L} e_{H \cup L}.$$

(3) For the left hook,

$$\lrcorner : \bigwedge^p E \times \bigwedge^{p+q} E^* \longrightarrow \bigwedge^q E^*,$$

we have

$$\begin{aligned} e_H \lrcorner e_L^* &= 0 \quad \textit{if } H \not\subseteq L \\ e_H \lrcorner e_L^* &= \rho_{L-H,H} e_{L-H}^* \quad \textit{if } H \subseteq L. \end{aligned}$$

Similar formulae hold for $\lrcorner : \bigwedge^p E^* \times \bigwedge^{p+q} E \longrightarrow \bigwedge^q E$. Using Proposition 22.26, we have the

Proposition 22.27 *For the left hook,*

$$\lrcorner : \bigwedge^p E \times \bigwedge^{p+q} E^* \longrightarrow \bigwedge^q E^*,$$

for every $u \in E$, we have

$$u \lrcorner (x^* \wedge y^*) = (-1)^s (u \lrcorner x^*) \wedge y^* + x^* \wedge (u \lrcorner y^*),$$

where $y \in \bigwedge^s E^*$.

Proof. We can prove the above identity assuming that x^* and y^* are of the form e_I^* and e_J^* using Proposition 22.26 but this is rather tedious. There is also a proof involving determinants, see Warner [147], Chapter 2. $\square$

Thus, $\lrcorner$ is almost an anti-derivation, except that the sign, $(-1)^s$ is applied to the wrong factor.

It is also possible to define a *right interior product or right hook*, $\llcorner$, using multiplication on the left rather than multiplication on the right. Then, $\llcorner$ defines a right action,

$$\llcorner : \bigwedge^{p+q} E^* \times \bigwedge^{p} E \longrightarrow \bigwedge^{q} E^*,$$

such that

$$\langle z^*, u \wedge v\rangle = \langle z^* \llcorner u, v\rangle, \qquad \text{for all } u \in \textstyle\bigwedge^p E,\ v \in \bigwedge^q E, \text{ and } z^* \in \bigwedge^{p+q} E^*.$$

Similarly, we have the right action

$$\llcorner : \bigwedge^{p+q} E \times \bigwedge^{p} E^* \longrightarrow \bigwedge^{q} E,$$

such that

$$\langle u^* \wedge v^*, z\rangle = \langle v^*, z \llcorner u^*\rangle, \qquad \text{for all } u^* \in \textstyle\bigwedge^p E^*,\ v^* \in \bigwedge^q E^*, \text{ and } z \in \bigwedge^{p+q} E.$$

Since the left hook, $\lrcorner : \bigwedge^p E \times \bigwedge^{p+q} E^* \longrightarrow \bigwedge^q E^*$, is defined by

$$\langle u \lrcorner z^*, v\rangle = \langle z^*, v \wedge u\rangle, \quad \text{for all } u \in \textstyle\bigwedge^p E,\ v \in \bigwedge^q E \text{ and } z^* \in \bigwedge^{p+q} E^*,$$

the right hook,

$$\llcorner : \bigwedge^{p+q} E^* \times \bigwedge^{p} E \longrightarrow \bigwedge^{q} E^*,$$

by

$$\langle z^* \llcorner u, v\rangle = \langle z^*, u \wedge v\rangle, \qquad \text{for all } u \in \textstyle\bigwedge^p E,\ v \in \bigwedge^q E, \text{ and } z^* \in \bigwedge^{p+q} E^*,$$

and $v \wedge u = (-1)^{pq} u \wedge v$, we conclude that

$$u \lrcorner z^* = (-1)^{pq} z^* \llcorner u,$$

where $u \in \bigwedge^p E$ and $z \in \bigwedge^{p+q} E^*$.

Using the above property and Proposition 22.27 we get the following version of Proposition 22.27 for the right hook:

Proposition 22.28 *For the right hook,*

$$\llcorner\colon \bigwedge^{p+q} E^* \times \bigwedge^{p} E \longrightarrow \bigwedge^{q} E^*,$$

for every $u \in E$, we have

$$(x^* \wedge y^*) \llcorner u = (x^* \llcorner u) \wedge y^* + (-1)^r x^* \wedge (y^* \llcorner u),$$

where $x^* \in \bigwedge^r E^*$.

Thus, $\llcorner$ is an anti-derivation.

For $u \in E$, the right hook, $z^* \llcorner u$, is also denoted, $i(u)z^*$, and called *insertion operator* or *interior product*. This operator plays an important role in differential geometry. If we view $z^* \in \bigwedge^{n+1}(E^*)$ as an alternating multilinear map in $\mathrm{Alt}^{n+1}(E; K)$, then $i(u)z^* \in \mathrm{Alt}^n(E; K)$ is given by

$$(i(u)z^*)(v_1, \ldots, v_n) = z^*(u, v_1, \ldots, v_n).$$

Note that certain authors, such as Shafarevitch [138], denote our right hook $z^* \llcorner u$ (which is also the right hook in Bourbaki [21] and Fulton and Harris [57]) by $u \lrcorner z^*$.

Using the two versions of $\lrcorner$, we can define linear maps $\gamma\colon \bigwedge^p E \to \bigwedge^{n-p} E^*$ and $\delta\colon \bigwedge^p E^* \to \bigwedge^{n-p} E$. For any basis $(e_1, \ldots, e_n)$ of E, if we let $M = \{1, \ldots, n\}$, $e = e_1 \wedge \cdots \wedge e_n$, and $e^* = e_1^* \wedge \cdots \wedge e_n^*$, then

$$\gamma(u) = u \lrcorner e^* \quad \text{and} \quad \delta(v) = v^* \lrcorner e,$$

for all $u \in \bigwedge^p E$ and all $v^* \in \bigwedge^p E^*$. The following proposition is easily shown.

Proposition 22.29 *The linear maps $\gamma\colon \bigwedge^p E \to \bigwedge^{n-p} E^*$ and $\delta\colon \bigwedge^p E^* \to \bigwedge^{n-p} E$ are isomorphims. The isomorphisms γ and δ map decomposable vectors to decomposable vectors. Furthermore, if $z \in \bigwedge^p E$ is decomposable, then $\langle \gamma(z), z\rangle = 0$, and similarly for $z \in \bigwedge^p E^*$. If $(e_1', \ldots, e_n')$ is any other basis of E and $\gamma'\colon \bigwedge^p E \to \bigwedge^{n-p} E^*$ and $\delta'\colon \bigwedge^p E^* \to \bigwedge^{n-p} E$ are the corresponding isomorphisms, then $\gamma' = \lambda\gamma$ and $\delta' = \lambda^{-1}\delta$ for some nonzero $\lambda \in \Omega$.*

Proof. Using Proposition 22.26, for any subset $J \subseteq \{1, \ldots, n\} = M$ such that $|J| = p$, we have

$$\gamma(e_J) = e_J \lrcorner e^* = \rho_{M-J,J} e^*_{M-J} \quad \text{and} \quad \delta(e^*_J) = e^*_J \lrcorner e = \rho_{M-J,J} e_{M-J}.$$

Thus,

$$\delta \circ \gamma(e_J) = \rho_{M-J,J}\rho_{J,M-J} e_J = (-1)^{p(n-p)} e_J.$$

A similar result holds for $\gamma \circ \delta$. This implies that

$$\delta \circ \gamma = (-1)^{p(n-p)}\mathrm{id} \quad \text{and} \quad \gamma \circ \delta = (-1)^{p(n-p)}\mathrm{id}.$$

Thus, γ and δ are isomorphisms. If $z \in \bigwedge^p E$ is decomposable, then $z = u_1 \wedge \cdots \wedge u_p$ where $u_1, \ldots, u_p$ are linearly independent since $z \neq 0$, and we can pick a basis of E of the form $(u_1, \ldots, u_n)$. Then, the above formulae show that

$$\gamma(z) = \pm u_{p+1}^* \wedge \cdots \wedge u_n^*.$$

Clearly

$$\langle \gamma(z), z \rangle = 0.$$

If $(e_1', \ldots, e_n')$ is any other basis of E, because $\bigwedge^m E$ has dimension 1, we have

$$e_1' \wedge \cdots \wedge e_n' = \lambda e_1 \wedge \cdots \wedge e_n$$

for some nonnull $\lambda \in \Omega$, and the rest is trivial. $\square$

We are now ready to tacke the problem of finding criteria for decomposability. We need a few preliminary results.

Proposition 22.30 *Given $z \in \bigwedge^p E$, with $z \neq 0$, the smallest vector space $W \subseteq E$ such that $z \in \bigwedge^p W$ is generated by the vectors of the form*

$$u^* \lrcorner z, \qquad \textit{with } u^* \in \bigwedge^{p-1} E^*.$$

Proof. First, let W be any subspace such that $z \in \bigwedge^p(E)$ and let $(e_1, \ldots, e_r, e_{r+1}, \ldots, e_n)$ be a basis of E such that $(e_1, \ldots, e_r)$ is a basis of W. Then, $u^* = \sum_I e_I^*$, where $I \subseteq \{1, \ldots, n\}$ and $|I| = p-1$, and $z = \sum_J e_J$, where $J \subseteq \{1, \ldots, r\}$ and $|J| = p \leq r$. It follows immediately from the formula of Proposition 22.26 (3) that $u^* \lrcorner z \in W$.

Next, we prove that if W is the smallest subspace of E such that $z \in \bigwedge^p(W)$, then W is generated by the vectors of the form $u^* \lrcorner z$, where $u^* \in \bigwedge^{p-1} E^*$. Suppose not, then the vectors $u^* \lrcorner z$ with $u^* \in \bigwedge^{p-1} E^*$ span a proper subspace, U, of W. We prove that for every subspace, W', of W, with $\dim(W') = \dim(W) - 1 = r - 1$, it is not possible that $u^* \lrcorner z \in W'$ for all $u^* \in \bigwedge^{p-1} E^*$. But then, as U is a proper subspace of W, it is contained in some subspace, W', with $\dim(W') = r - 1$ and we have a contradiction.

Let $w \in W - W'$ and pick a basis of W formed by a basis $(e_1, \ldots, e_{r-1})$ of W' and w. We can write $z = z' + w \wedge z''$, where $z' \in \bigwedge^p W'$ and $z'' \in \bigwedge^{p-1} W'$, and since W is the smallest subspace containing z, we have $z'' \neq 0$. Consequently, if we write $z'' = \sum_I e_I$ in terms of the basis $(e_1, \ldots, e_{r-1})$ of W', there is some e_I, with $I \subseteq \{1, \ldots, r-1\}$ and $|I| = p - 1$, so that the coefficient λ_I is nonzero. Now, using any basis of E containing $(e_1, \ldots, e_{r-1}, w)$, by Proposition 22.26 (3), we see that

$$e_I^* \lrcorner (w \wedge e_I) = \lambda w, \qquad \lambda = \pm 1.$$

It follows that

$$e_I^* \lrcorner z = e_I^* \lrcorner (z' + w \wedge z'') = e_I^* \lrcorner z' + e_I^* \lrcorner (w \wedge z'') = e_I^* \lrcorner z' + \lambda w,$$

with $e_I^* \lrcorner z' \in W'$, which shows that $e_I^* \lrcorner z \notin W'$. Therefore, W is indeed generated by the vectors of the form $u^* \lrcorner z$, where $u^* \in \bigwedge^{p-1} E^*$. $\square$

Proposition 22.31 *Any nonzero* $z \in \bigwedge^p E$ *is decomposable iff*

$$(u^* \lrcorner z) \wedge z = 0, \qquad \textit{for all } u^* \in \textstyle\bigwedge^{p-1} E^*.$$

Proof. Clearly, $z \in \bigwedge^p E$ is decomposable iff the smallest vector space, W, such that $z \in \bigwedge^p W$ has dimension p. If $\dim(W) = p$, we have $z = e_1 \wedge \cdots \wedge e_p$ where $e_1, \ldots, e_p$ form a basis of W. By Proposition 22.30, for every $u^* \in \bigwedge^{p-1} E^*$, we have $u^* \lrcorner z \in W$, so each $u^* \lrcorner z$ is a linear combination of the e_i's and $(u^* \lrcorner z) \wedge z = (u^* \lrcorner z) \wedge e_1 \wedge \cdots \wedge e_p = 0$.

Now, assume that $(u^* \lrcorner z) \wedge z = 0$ for all $u^* \in \bigwedge^{p-1} E^*$ and that $\dim(W) = n > p$. If $e_1, \ldots, e_n$ is a basis of W, then we have $z = \sum_I \lambda_I e_I$, where $I \subseteq \{1, \ldots, n\}$ and $|I| = p$. Recall that $z \neq 0$, and so, some λ_I is nonzero. By Proposition 22.30, each e_i can be written as $u^* \lrcorner z$ for some $u^* \in \bigwedge^{p-1} E^*$ and since $(u^* \lrcorner z) \wedge z = 0$ for all $u^* \in \bigwedge^{p-1} E^*$, we get

$$e_j \wedge z = 0 \quad \text{for} \quad j = 1, \ldots, n.$$

By wedging $z = \sum_I \lambda_I e_I$ with each e_j, as $n > p$, we deduce $\lambda_I = 0$ for all I, so $z = 0$, a contradiction. Therefore, $n = p$ and z is decomposable. $\square$

In Proposition 22.31, we can let u^* range over a basis of $\bigwedge^{p-1} E^*$, and then, the conditions are

$$(e_H^* \lrcorner z) \wedge z = 0$$

for all $H \subseteq \{1, \ldots, n\}$, with $|H| = p - 1$. Since $(e_H^* \lrcorner z) \wedge z \in \bigwedge^{p+1} E$, this is equivalent to

$$e_J^* \left((e_H^* \lrcorner z) \wedge z\right) = 0$$

for all $H, J \subseteq \{1, \ldots, n\}$, with $|H| = p - 1$ and $|J| = p + 1$. Then, for all $I, I' \subseteq \{1, \ldots, n\}$ with $|I| = |I'| = p$, we can show that

$$e_J^* \left((e_H^* \lrcorner e_I) \wedge e_{I'}\right) = 0,$$

unless there is some $i \in \{1, \ldots, n\}$ such that

$$I - H = \{i\}, \quad J - I' = \{i\}.$$

In this case,

$$e_J^* \left((e_H^* \lrcorner e_{H \cup \{i\}}) \wedge e_{J - \{i\}}\right) = \rho_{\{i\},H}\, \rho_{\{i\},J-\{i\}}.$$

If we let

$$\epsilon_{i,J,H} = \rho_{\{i\},H}\, \rho_{\{i\},J-\{i\}},$$

we have $\epsilon_{i,J,H} = +1$ if the parity of the number of $j \in J$ such that $j < i$ is the same as the parity of the number of $h \in H$ such that $h < i$, and $\epsilon_{i,J,H} = -1$ otherwise.

Finally, we obtain the following criterion in terms of quadratic equations (*Plücker's equations*) for the decomposability of an alternating tensor:

Proposition 22.32 *(Grassmann-Plücker's Equations) For $z = \sum_I \lambda_I e_I \in \bigwedge^p E$, the conditions for $z \neq 0$ to be decomposable are*

$$\sum_{i \in J-H} \epsilon_{i,J,H} \lambda_{H\cup\{i\}} \lambda_{J-\{i\}} = 0,$$

for all $H, J \subseteq \{1, \ldots, n\}$ such that $|H| = p-1$ and $|J| = p+1$.

Using these criteria, it is a good exercise to prove that if $\dim(E) = n$, then every tensor in $\bigwedge^{n-1}(E)$ is decomposable. This can also be shown directly.

It should be noted that the equations given by Proposition 22.32 are not independent. For example, when $\dim(E) = n = 4$ and $p = 2$, these equations reduce to the single equation

$$\lambda_{12}\lambda_{34} - \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23} = 0.$$

When the field, K, is the field of complex numbers, this is the homogeneous equation of a quadric in $\mathbb{CP}^5$ known as the *Klein quadric*. The points on this quadric are in one-to-one correspondence with the lines in $\mathbb{CP}^3$.

22.18 Vector-Valued Alternating Forms

In this section, the vector space, E, is assumed to have finite dimension. We know that there is a canonical isomorphism, $\bigwedge^n(E^*) \cong \mathrm{Alt}^n(E; K)$, between alternating n-forms and alternating multilinear maps. As in the case of general tensors, the isomorphisms,

$$\begin{aligned}
\mathrm{Alt}^n(E; F) &\cong \mathrm{Hom}(\bigwedge^n(E), F) \\
\mathrm{Hom}(\bigwedge^n(E), F) &\cong (\bigwedge^n(E))^* \otimes F \\
(\bigwedge^n(E))^* &\cong \bigwedge^n(E^*)
\end{aligned}$$

yield a canonical isomorphism

$$\mathrm{Alt}^n(E; F) \cong \left(\bigwedge^n(E^*)\right) \otimes F.$$

Note that F may have infinite dimension. This isomorphism allows us to view the tensors in $\bigwedge^n(E^*) \times F$ as *vector valued alternating forms*, a point of view that is useful in differential geometry. If $(f_1, \ldots, f_r)$ is a basis of F, every tensor, $\omega \in \bigwedge^n(E^*) \times F$ can be written as some linear combination

$$\omega = \sum_{i=1}^{r} \alpha_i \otimes f_i,$$

with $\alpha_i \in \bigwedge^n(E^*)$. We also let

$$\bigwedge(E;F) = \bigoplus_{n=0} \left(\bigwedge^n(E^*)\right) \otimes F = \left(\bigwedge(E)\right) \otimes F.$$

Given three vector spaces, F, G, H, if we have some bilinear map, $\Phi\colon F \otimes G \to H$, then we can define a multiplication operation,

$$\wedge_\Phi\colon \bigwedge(E;F) \times \bigwedge(E;G) \to \bigwedge(E;H),$$

as follows: For every pair, (m,n), we define the multiplication,

$$\wedge_\Phi\colon \left(\left(\bigwedge^m(E^*)\right) \otimes F\right) \times \left(\left(\bigwedge^n(E^*)\right) \otimes G\right) \longrightarrow \left(\bigwedge^{m+n}(E^*)\right) \otimes H,$$

by

$$(\alpha \otimes f) \wedge_\Phi (\beta \otimes g) = (\alpha \wedge \beta) \otimes \Phi(f,g).$$

As in Section 22.15 (following H. Cartan [30]) we can also define a multiplication,

$$\wedge_\Phi\colon \mathrm{Alt}^m(E;F) \times \mathrm{Alt}^m(E;G) \longrightarrow \mathrm{Alt}^{m+n}(E;H),$$

directly on alternating multilinear maps as follows: For $f \in \mathrm{Alt}^m(E;F)$ and $g \in \mathrm{Alt}^n(E;G)$,

$$(f \wedge_\Phi g)(u_1, \ldots, u_{m+n}) = \sum_{\sigma \in \mathrm{shuffle}(m,n)} \mathrm{sgn}(\sigma)\, \Phi(f(u_{\sigma(1)}, \ldots, u_{\sigma(m)}), g(u_{\sigma(m+1)}, \ldots, u_{\sigma(m+n)})),$$

where $\mathrm{shuffle}(m,n)$ consists of all (m,n)-"shuffles", that is, permutations, σ, of $\{1, \ldots m+n\}$, such that $\sigma(1) < \cdots < \sigma(m)$ and $\sigma(m+1) < \cdots < \sigma(m+n)$.

In general, not much can be said about $\wedge_\Phi$ unless Φ has some additional properties. In particular, $\wedge_\Phi$ is generally not associative. We also have the map,

$$\mu\colon \left(\bigwedge^n(E^*)\right) \otimes F \longrightarrow \mathrm{Alt}^n(E;F),$$

defined on generators by

$$\mu((v_1^* \wedge \cdots \wedge v_n^*) \otimes a)(u_1, \ldots, u_n) = (\det(v_j^*(u_i))a.$$

Proposition 22.33 *The map*

$$\mu\colon \left(\bigwedge^n(E^*)\right) \otimes F \longrightarrow \mathrm{Alt}^n(E;F),$$

defined as above is a canonical isomorphism for every $n \geq 0$. Furthermore, given any three vector spaces, F, G, H, and any bilinear map, $\Phi\colon F \times G \to H$, for all $\omega \in (\bigwedge^n(E^*)) \otimes F$ and all $\eta \in (\bigwedge^n(E^*)) \otimes G$,

$$\mu(\alpha \wedge_\Phi \beta) = \mu(\alpha) \wedge_\Phi \mu(\beta).$$

Proof. Since we already know that $(\bigwedge^n(E^*))\otimes F$ and $\mathrm{Alt}^n(E;F)$ are isomorphic, it is enough to show that μ maps some basis of $(\bigwedge^n(E^*))\otimes F$ to linearly independent elements. Pick some bases, $(e_1,\ldots,e_p)$ in E and $(f_j)_{j\in J}$ in F. Then, we know that the vectors, $e_I^*\otimes f_j$, where $I\subseteq\{1,\ldots,p\}$ and $|I|=n$ form a basis of $(\bigwedge^n(E^*))\otimes F$. If we have a linear dependence,

$$\sum_{I,j}\lambda_{I,j}\mu(e_I^*\otimes f_j)=0,$$

applying the above combination to each $(e_{i_1},\ldots,e_{i_n})$ ($I=\{i_1,\ldots,i_n\}$, $i_1<\cdots<i_n$), we get the linear combination

$$\sum_j\lambda_{I,j}f_j=0,$$

and by linear independence of the f_j's, we get $\lambda_{I,j}=0$, for all I and all j. Therefore, the $\mu(e_I^*\otimes f_j)$ are linearly independent and we are done. The second part of the proposition is easily checked (a simple computation). □

A special case of interest is the case where $F=G=H$ is a Lie algebra and $\Phi(a,b)=[a,b]$, is the Lie bracket of F. In this case, using a base, $(f_1,\ldots,f_r)$, of F if we write $\omega=\sum_i\alpha_i\otimes f_i$ and $\eta=\sum_j\beta_j\otimes f_j$, we have

$$[\omega,\eta]=\sum_{i,j}\alpha_i\wedge\beta_j\otimes[f_i,f_j].$$

Consequently,

$$[\eta,\omega]=(-1)^{mn+1}[\omega,\eta].$$

The following proposition will be useful in dealing with vector-valued differential forms:

Proposition 22.34 *If $(e_1,\ldots,e_p)$ is any basis of E, then every element, $\omega\in(\bigwedge^n(E^*))\otimes F$, can be written in a unique way as*

$$\omega=\sum_I e_I^*\otimes f_I,\qquad f_I\in F,$$

where the e_I^* are defined as in Section 22.12.

Proof. Since, by Proposition 22.19, the e_I^* form a basis of $\bigwedge^n(E^*)$, elements of the form $e_I^*\otimes f$ span $(\bigwedge^n(E^*))\otimes F$. Now, if we apply $\mu(\omega)$ to $(e_{i_1},\ldots,e_{i_n})$, where $I=\{i_1,\ldots,i_n\}\subseteq\{1,\ldots,p\}$, we get

$$\mu(\omega)(e_{i_1},\ldots,e_{i_n})=\mu(e_I^*\otimes f_I)(e_{i_1},\ldots,e_{i_n})=f_I.$$

Therefore, the f_I are uniquely determined by f. □

Proposition can also be formulated in terms of alternating multilinear maps, a fact that will be useful to deal with differential forms.

Define the product, $\cdot\colon \mathrm{Alt}^n(E;\mathbb{R}) \times F \to \mathrm{Alt}^n(E;F)$, as follows: For all $\omega \in \mathrm{Alt}^n(E;\mathbb{R})$ and all $f \in F$,

$$(\omega \cdot f)(u_1, \ldots, u_n) = \omega(u_1, \ldots, u_n)f,$$

for all $u_1, \ldots, u_n \in E$. Then, it is immediately verified that for every $\omega \in (\bigwedge^n(E^*)) \otimes F$ of the form

$$\omega = u_1^* \wedge \cdots \wedge u_n^* \otimes f,$$

we have

$$\mu(u_1^* \wedge \cdots \wedge u_n^* \otimes f) = \mu(u_1^* \wedge \cdots \wedge u_n^*) \cdot f.$$

Then, Proposition 22.34 yields

Proposition 22.35 *If $(e_1, \ldots, e_p)$ is any basis of E, then every element, $\omega \in \mathrm{Alt}^n(E;F)$, can be written in a unique way as*

$$\omega = \sum_I e_I^* \cdot f_I, \qquad f_I \in F,$$

where the e_I^* are defined as in Section 22.12.

22.19 Tensor Products of Modules over a Commmutative Ring

If R is a commutative ring with identity (say 1), recall that a *module over R* (or *R-module*) is an abelian group, M, with a scalar multiplication, $\cdot\colon R \times M \to M$, and all the axioms of a vector space are satisfied.

At first glance, a module does not seem any different from a vector space but the lack of multiplicative inverses in R has drastic consequences, one being that unlike vector spaces, modules are generally not free, that is, have no bases. Furthermore, a module may have *torsion elements*, that is, elements, $m \in M$, such that $\lambda \cdot m = 0$, even though $m \neq 0$ and $\lambda \neq 0$.

Nevertheless, it is possible to define tensor products of modules over a ring, just as in Section 22.1 and the results of this section continue to hold. The results of Section 22.3 also continue to hold since they are based on the universal mapping property. However, the results of Section 22.2 on bases generally fail, except for free modules. Similarly, the results of Section 22.4 on duality generally fail. Tensor algebras can be defined for modules, as in Section 22.5. Symmetric tensor and alternating tensors can be defined for modules but again, results involving bases generally fail.

Tensor products of modules have some unexpected properties. For example, if p and q are relatively prime integers, then

$$\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/q\mathbb{Z} = (0).$$

It is possible to salvage certain properties of tensor products holding for vector spaces by restricting the class of modules under consideration. For example, *projective modules*, have a pretty good behavior w.r.t. tensor products.

A free R-module, F, is a module that has a basis (*i.e.*, there is a family, $(e_i)_{i\in I}$, of linearly independent vectors in F that span F). Projective modules have many equivalent characterizations. Here is one that is best suited for our needs:

Definition 22.9 An R-module, P, is *projective* if it is a summand of a free module, that is, if there is a free R-module, F, and some R-module, Q, so that

$$F = P \oplus Q.$$

Given any R-module, M, we let $M^* = \mathrm{Hom}_R(M, R)$ be its *dual*. We have the following proposition:

Proposition 22.36 *For any finitely-generated projective R-modules, P, and any R-module, Q, we have the isomorphisms:*

$$\begin{aligned} P^{**} &\cong P \\ \mathrm{Hom}_R(P, Q) &\cong P^* \otimes_R Q. \end{aligned}$$

Sketch of proof. We only consider the second isomorphism. Since P is projective, we have some R-modules, P_1, F, with

$$P \oplus P_1 = F,$$

where F is some free module. Now, we know that for any R-modules, U, V, W, we have

$$\mathrm{Hom}_R(U \oplus V, W) \cong \mathrm{Hom}_R(U, W) \prod \mathrm{Hom}_R(V, W) \cong \mathrm{Hom}_R(U, W) \oplus \mathrm{Hom}_R(V, W),$$

so

$$P^* \oplus P_1^* \cong F^*, \qquad \mathrm{Hom}_R(P, Q) \oplus \mathrm{Hom}_R(P_1, Q) \cong \mathrm{Hom}_R(F, Q).$$

By tensoring with Q and using the fact that tensor distributes w.r.t. coproducts, we get

$$(P^* \otimes_R Q) \oplus (P_1^* \otimes Q) \cong (P^* \oplus P_1^*) \otimes_R Q \cong F^* \otimes_R Q.$$

Now, the proof of Proposition 22.9 goes through because F is free and finitely generated, so

$$\alpha_\otimes \colon (P^* \otimes_R Q) \oplus (P_1^* \otimes Q) \cong F^* \otimes_R Q \longrightarrow \mathrm{Hom}_R(F, Q) \cong \mathrm{Hom}_R(P, Q) \oplus \mathrm{Hom}_R(P_1, Q)$$

is an isomorphism and as α_α maps $P^* \otimes_R Q$ to $\mathrm{Hom}_R(P, Q)$, it yields an isomorphism between these two spaces. $\square$

The isomorphism $\alpha_\otimes \colon P^* \otimes_R Q \cong \mathrm{Hom}_R(P, Q)$ of Proposition 22.36 is still given by

$$\alpha_\otimes(u^* \otimes f)(x) = u^*(x) f, \qquad u^* \in P^*,\ f \in Q,\ x \in P.$$

It is convenient to introduce the *evaluation map*, $\mathrm{Ev}_x\colon P^* \otimes_R Q \to Q$, defined for every $x \in P$ by

$$\mathrm{Ev}_x(u^* \otimes f) = u^*(x)f, \qquad u^* \in P^*,\ f \in Q.$$

In Section 11.2 we will need to consider a slightly weaker version of the universal mapping property of tensor products. The situation is this: We have a commutative R-algebra, S, where R is a field (or even a commutative ring), we have two R-modules, U and V, and moreover, U is a right S-module and V is a left S-module. In Section 11.2, this corresponds to $R = \mathbb{R}$, $S = C^\infty(B)$, $U = \mathcal{A}^i(\xi)$ and $V = \Gamma(\xi)$, where ξ is a vector bundle. Then, we can form the tensor product, $U \otimes_R V$, and we let $U \otimes_S V$ be the quotient module, $(U \otimes_R V)/W$, where W is the submodule of $U \otimes_R V$ generated by the elements of the form

$$us \otimes_R v - u \otimes_R sv.$$

As S is commutative, we can make $U \otimes_S V$ into an S-module by defining the action of S *via*

$$s(u \otimes_S v) = us \otimes_S v.$$

It is immediately verified that this S-module is isomorphic to the tensor product of U and V as S-modules and the following universal mapping property holds:

Proposition 22.37 *For every, R-bilinear map, $f\colon U \times V \to Z$, if f satisfies the property*

$$f(us, v) = f(u, sv), \qquad \text{for all}\, u \in U,\ v \in V,\ s \in S,$$

then f induces a unique R-linear map, $\widehat{f}\colon U \otimes_S V \to Z$, such that

$$f(u, v) = \widehat{f}(u \otimes_S v), \qquad \text{for all } u \in U,\ v \in V.$$

Note that the linear map, $\widehat{f}\colon U \otimes_S V \to Z$, is only R-linear, it is not S-linear in general.

22.20 The Pfaffian Polynomial

Let $\mathfrak{so}(2n)$ denote the vector space (actually, Lie algebra) of $2n \times 2n$ real skew-symmetric matrices. It is well-known that every matrix, $A \in \mathfrak{so}(2n)$, can be written as

$$A = PDP^\top,$$

where P is an orthogonal matrix and where D is a block diagonal matrix

$$D = \begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_n \end{pmatrix}$$

consisting of 2×2 blocks of the form

$$D_i = \begin{pmatrix} 0 & -a_i \\ a_i & 0 \end{pmatrix}.$$

For a proof, see see Horn and Johnson [79], Corollary 2.5.14, Gantmacher [61], Chapter IX, or Gallier [58], Chapter 11.

Since $\det(D_i) = a_i^2$ and $\det(A) = \det(PDP^\top) = \det(D) = \det(D_1) \cdots \det(D_n)$, we get

$$\det(A) = (a_1 \cdots a_n)^2.$$

The Pfaffian is a polynomial function, $\mathrm{Pf}(A)$, in skew-symmetric $2n \times 2n$ matrices, A, (a polynomial in $(2n-1)n$ variables) such that

$$\mathrm{Pf}(A)^2 = \det(A)$$

and for every arbitrary matrix, B,

$$\mathrm{Pf}(BAB^\top) = \mathrm{Pf}(A)\det(B).$$

The Pfaffian shows up in the definition of the Euler class of a vector bundle. There is a simple way to define the Pfaffian using some exterior algebra. Let $(e_1, \ldots, e_{2n})$ be any basis of $\mathbb{R}^{2n}$. For any matrix, $A \in \mathfrak{so}(2n)$, let

$$\omega(A) = \sum_{i<j} a_{ij}\, e_i \wedge e_j,$$

where $A = (a_{ij})$. Then, $\bigwedge^n \omega(A)$ is of the form $Ce_1 \wedge e_2 \wedge \cdots \wedge e_{2n}$ for some constant, $C \in \mathbb{R}$.

Definition 22.10 For every skew symmetric matrix, $A \in \mathfrak{so}(2n)$, the *Pfaffian polynomial* or *Pfaffian* is the degree n polynomial, $\mathrm{Pf}(A)$, defined by

$$\bigwedge^n \omega(A) = n!\, \mathrm{Pf}(A)\, e_1 \wedge e_2 \wedge \cdots \wedge e_{2n}.$$

Clearly, $\mathrm{Pf}(A)$ is independent of the basis chosen. If A is the block diagonal matrix D, a simple calculation shows that

$$\omega(D) = -(a_1 e_1 \wedge e_2 + a_2 e_3 \wedge e_4 + \cdots + a_n e_{2n-1} \wedge e_{2n})$$

and that

$$\bigwedge^n \omega(D) = (-1)^n n!\, a_1 \cdots a_n\, e_1 \wedge e_2 \wedge \cdots \wedge e_{2n},$$

and so

$$\mathrm{Pf}(D) = (-1)^n a_1 \cdots a_n.$$

Since $\mathrm{Pf}(D)^2 = (a_1 \cdots a_n)^2 = \det(A)$, we seem to be on the right track.

Proposition 22.38 *For every skew symmetric matrix, $A \in \mathfrak{so}(2n)$ and every arbitrary matrix, B, we have:*

(i) $\mathrm{Pf}(A)^2 = \det(A)$

(ii) $\mathrm{Pf}(BAB^\top) = \mathrm{Pf}(A)\det(B)$.

Proof. If we assume that (ii) is proved then, since we can write $A = PDP^\top$ for some orthogonal matrix, P, and some block diagonal matrix, D, as above, as $\det(P) = \pm 1$ and $\mathrm{Pf}(D)^2 = \det(A)$, we get

$$\mathrm{Pf}(A)^2 = \mathrm{Pf}(PDP^\top)^2 = \mathrm{Pf}(D)^2 \det(P)^2 = \det(A),$$

which is (i). Therefore, it remains to prove (ii).

Let $f_i = Be_i$, for $i = 1, \ldots, 2n$, where $(e_1, \ldots, e_{2n})$ is any basis of $\mathbb{R}^{2n}$. Since $f_i = \sum_k b_{ki} e_k$, we have

$$\tau = \sum_{i,j} a_{ij}\, f_i \wedge f_j = \sum_{i,j} \sum_{k,l} b_{ki} a_{ij} b_{lj}\, e_k \wedge e_l = \sum_{k,l} (BAB^\top)_{kl}\, e_k \wedge e_l,$$

and so, as $BAB^\top$ is skew symmetric and $e_k \wedge e_l = -e_l \wedge e_k$, we get

$$\tau = 2\omega(BAB^\top).$$

Consequently,

$$\bigwedge^n \tau = 2^n \bigwedge^n \omega(BAB^\top) = 2^n n!\, \mathrm{Pf}(BAB^\top)\, e_1 \wedge e_2 \wedge \cdots \wedge e_{2n}.$$

Now,

$$\bigwedge^n \tau = C\, f_1 \wedge f_2 \wedge \cdots \wedge f_{2n},$$

for some $C \in \mathbb{R}$. If B is singular, then the f_i are linearly dependent which implies that $f_1 \wedge f_2 \wedge \cdots \wedge f_{2n} = 0$, in which case,

$$\mathrm{Pf}(BAB^\top) = 0,$$

as $e_1 \wedge e_2 \wedge \cdots \wedge e_{2n} \neq 0$. Therefore, if B is singular, $\det(B) = 0$ and

$$\mathrm{Pf}(BAB^\top) = 0 = \mathrm{Pf}(A)\det(B).$$

If B is invertible, as $\tau = \sum_{i,j} a_{ij}\, f_i \wedge f_j = 2\sum_{i<j} a_{ij}\, f_i \wedge f_j$, we have

$$\bigwedge^n \tau = 2^n n!\, \mathrm{Pf}(A)\, f_1 \wedge f_2 \wedge \cdots \wedge f_{2n}.$$

However, as $f_i = Be_i$, we have

$$f_1 \wedge f_2 \wedge \cdots \wedge f_{2n} = \det(B)\, e_1 \wedge e_2 \wedge \cdots \wedge e_{2n},$$

so

$$\bigwedge^{n} \tau = 2^n n! \,\mathrm{Pf}(A) \det(B)\, e_1 \wedge e_2 \wedge \cdots \wedge e_{2n}$$

and as

$$\bigwedge^{n} \tau = 2^n n! \,\mathrm{Pf}(BAB^\top)\, e_1 \wedge e_2 \wedge \cdots \wedge e_{2n},$$

we get

$$\mathrm{Pf}(BAB^\top) = \mathrm{Pf}(A) \det(B),$$

as claimed. $\square$

Remark: It can be shown that the polynomial, $\mathrm{Pf}(A)$, is the unique polynomial with integer coefficients such that $\mathrm{Pf}(A)^2 = \det(A)$ and $\mathrm{Pf}(\mathrm{diag}(S, \ldots, S)) = +1$, where

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

see Milnor and Stasheff [110] (Appendix C, Lemma 9). There is also an explicit formula for $\mathrm{Pf}(A)$, namely:

$$\mathrm{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1)\, \sigma(2i)}.$$

Beware, some authors use a different sign convention and require the Pfaffian to have the value $+1$ on the matrix $\mathrm{diag}(S', \ldots, S')$, where

$$S' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For example, if $\mathbb{R}^{2n}$ is equipped with an inner product, $\langle -, - \rangle$, then some authors define $\omega(A)$ as

$$\omega(A) = \sum_{i<j} \langle Ae_i, e_j \rangle\, e_i \wedge e_j,$$

where $A = (a_{ij})$. But then, $\langle Ae_i, e_j \rangle = a_{ji}$ and **not** a_{ij}, and this Pfaffian takes the value $+1$ on the matrix $\mathrm{diag}(S', \ldots, S')$. This version of the Pfaffian differs from our version by the factor $(-1)^n$. In this respect, Madsen and Tornehave [100] seem to have an incorrect sign in Proposition B6 of Appendix C.

We will also need another property of Pfaffians. Recall that the ring, $M_n(\mathbb{C})$, of $n \times n$ matrices over $\mathbb{C}$ is embedded in the ring, $M_{2n}(\mathbb{R})$, of $2n \times 2n$ matrices with real coefficients, using the injective homomorphism that maps every entry $z = a + ib \in \mathbb{C}$ to the 2×2 matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

If $A \in M_n(\mathbb{C})$, let $A_{\mathbb{R}} \in M_{2n}(\mathbb{R})$ denote the real matrix obtained by the above process. Observe that every skew Hermitian matrix, $A \in \mathfrak{u}(n)$, (*i.e.*, with $A^* = \overline{A}^\top = -A$) yields a matrix $A_{\mathbb{R}} \in \mathfrak{so}(2n)$.

Proposition 22.39 *For every skew Hermitian matrix, $A \in \mathfrak{u}(n)$, we have*

$$\mathrm{Pf}(A_{\mathbb{R}}) = i^n \det(A).$$

Proof. It is well-known that a skew Hermitian matrix can be diagonalized with respect to a unitary matrix, U, and that the eigenvalues are pure imaginary or zero, so we can write

$$A = U \,\mathrm{diag}(ia_1, \ldots, ia_n)U^*,$$

for some reals, $a_i \in \mathbb{R}$. Consequently, we get

$$A_{\mathbb{R}} = U_{\mathbb{R}} \,\mathrm{diag}(D_1, \ldots, D_n)U_{\mathbb{R}}^{\top},$$

where

$$D_i = \begin{pmatrix} 0 & -a_i \\ a_i & 0 \end{pmatrix}$$

and

$$\mathrm{Pf}(A_{\mathbb{R}}) = \mathrm{Pf}(\mathrm{diag}(D_1, \ldots, D_n)) = (-1)^n \, a_1 \cdots a_n,$$

as we saw before. On the other hand,

$$\det(A) = \det(\mathrm{diag}(ia_1, \ldots, ia_n)) = i^n \, a_1 \cdots a_n,$$

and as $(-1)^n = i^n i^n$, we get

$$\mathrm{Pf}(A_{\mathbb{R}}) = i^n \det(A),$$

as claimed. $\square$

Madsen and Tornehave [100] state Proposition 22.39 using the factor $(-i)^n$, which is wrong.

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