Chapter 17

Discrete Laplacians on Polyhedral Surfaces

Chapter 18

Metrics and Curvature on Lie Groups

18.1 Left (resp. Right) Invariant Metrics

Since a Lie group, G, is a smooth manifold, we can endow G with a Riemannian metric. Among all the Riemannian metrics on a Lie groups, those for which the left translations (or the right translations) are isometries are of particular interest because they take the group structure of G into account. As a consequence, it is possible to find explicit formulae for the Levi-Civita connection and the various curvatures, especially in the case of metrics which are both left and right-invariant. This chapter makes extensive use of results from a beautiful paper of Milnor [109].

Definition 18.1 A metric, $\langle -, - \rangle$, on a Lie group, G, is called *left-invariant* (resp. *right-invariant*) iff

$$\langle u, v\rangle_b = \langle (dL_a)_b u, (dL_a)_b v\rangle_{ab},$$

(resp.

$$\langle u, v\rangle_b = \langle (dR_a)_b u, (dR_a)_b v\rangle_{ba}),$$

or all $a, b \in G$ and all $u, v \in T_bG$. A Riemannian metric that is both left and right-invariant is called a *bi-invariant metric*.

As shown in the next proposition, left-invariant (resp. right-invariant) metrics on G are induced by inner products on the Lie algebra, $\mathfrak{g}$, of G. In the sequel, the identity element of the Lie group, G, will be denoted by e or 1.

Proposition 18.1 *There is a bijective correspondence between left-invariant (resp. right invariant) metrics on a Lie group, G, and inner products on the Lie algebra, $\mathfrak{g}$, of G.*

Proof. If the metric on G is left-invariant, then for all $a \in G$ and all $u, v \in T_aG$, we have

$$\begin{aligned} \langle u, v\rangle_a &= \langle d(L_a \circ L_{a^{-1}})_a u, d(L_a \circ L_{a^{-1}})_a v\rangle_a \\ &= \langle (dL_a)_e((dL_{a^{-1}})_a u), (dL_a)_e((dL_{a^{-1}})_a v)\rangle_a \\ &= \langle (dL_{a^{-1}})_a u, (dL_{a^{-1}})_a v\rangle_e, \end{aligned}$$

which shows that our metric is completely determined by its restriction to $\mathfrak{g} = T_eG$. Conversely, let $\langle -, -\rangle$ be an inner product on $\mathfrak{g}$ and set

$$\langle u, v\rangle_g = \langle (dL_{g^{-1}})_g u, (dL_{g^{-1}})_g v\rangle,$$

for all $u, v \in T_gG$ and all $g \in G$. Obviously, the family of inner products, $\langle -, -\rangle_g$, yields a Riemannian metric on G. To prove that it is left-invariant, we use the chain rule and the fact that left translations are group isomorphisms. For all $a, b \in G$ and all $u, v \in T_bG$, we have

$$\begin{aligned}
\langle (dL_a)_b u, (dL_a)_b v\rangle_{ab} &= \langle (dL_{(ab)^{-1}})_{ab}((dL_a)_b u), (dL_{(ab)^{-1}})_{ab}((dL_a)_b v)\rangle \\
&= \langle d(L_{(ab)^{-1}} \circ L_a)_b u, d(L_{(ab)^{-1}} \circ L_a)_b v\rangle \\
&= \langle d(L_{b^{-1}a^{-1}} \circ L_a)_b u, d(L_{b^{-1}a^{-1}} \circ L_a)_b v\rangle \\
&= \langle (dL_{b^{-1}})_b u, (dL_{b^{-1}})_b v\rangle \\
&= \langle u, v\rangle_b,
\end{aligned}$$

as desired.

To get a right-invariant metric on G, set

$$\langle u, v\rangle_g = \langle (dR_{g^{-1}})_g u, (dR_{g^{-1}})_g v\rangle,$$

for all $u, v \in T_gG$ and all $g \in G$. The verification that this metric is right-invariant is analogous. □

If G has dimension n, then since inner products on $\mathfrak{g}$ are in one-to-one correspondence with $n \times n$ positive definite matrices, we see that G possesses a family of left-invariant metrics of dimension $\frac{1}{2}\, n(n+1)$.

If G has a left-invariant (resp. right-invariant) metric, since left-invariant (resp. right-invariant) translations are isometries and act transitively on G, the space G is called a *homogeneous Riemannian manifold.*

Proposition 18.2 *Every Lie group, G, equipped with a left-invariant (resp. right-invariant) metric is complete.*

Proof. As G is locally compact, we can pick some $\epsilon > 0$ small enough so that the closed ϵ-ball about the identity is compact. By translation, every ϵ-ball is compact, hence every Cauchy sequence eventually lies within a compact set and thus, converges. □

We now give several characterizations of bi-invariant metrics.

18.2 Bi-Invariant Metrics

Recall that the adjoint representation, $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$, is the map defined such that $\mathrm{Ad}_a\colon \mathfrak{g} \to \mathfrak{g}$ is the linear isomorphism given by

$$\mathrm{Ad}_a = d(R_{a^{-1}} \circ L_a)_e,$$

for every $a \in G$. Clearly,

$$\mathrm{Ad}_a = (dR_{a^{-1}})_a \circ (dL_a)_e.$$

Here is the first of four criteria for the existence of a bi-invariant metric on a Lie group.

Proposition 18.3 *There is a bijective correspondence between bi-invariant metrics on a Lie group, G, and* Ad*-invariant inner products on the Lie algebra, $\mathfrak{g}$, of G, that is, inner products, $\langle -, - \rangle$, on $\mathfrak{g}$ such that Ad_a is an isometry of $\mathfrak{g}$ for all $a \in G$; more explicitly, inner products such that*

$$\langle \mathrm{Ad}_a u, \mathrm{Ad}_a v \rangle = \langle u, v \rangle,$$

for all $a \in G$ and all $u, v \in \mathfrak{g}$.

Proof. If $\langle -, - \rangle$ is a bi-invariant metric on G, as

$$\mathrm{Ad}_a = (dR_{a^{-1}})_a \circ (dL_a)_e.$$

it is clear that Ad_a is an isometry on $\mathfrak{g}$.

Conversely, if $\langle -, - \rangle$ is any inner product on $\mathfrak{g}$ such that Ad_a is an isometry of $\mathfrak{g}$ for all $a \in G$, we need to prove that the metric on G given by

$$\langle u, v \rangle_g = \langle (dL_{g^{-1}})_g u, (dL_{g^{-1}})_g v \rangle$$

is also right-invariant. We have

$$\begin{aligned}
\langle (dR_a)_b u, (dR_a)_b v \rangle_{ba} &= \langle (dL_{(ba)^{-1}})_{ba}((dR_a)_b u), (dL_{(ba)^{-1}})_{ba}((dR_a)_b v) \rangle \\
&= \langle d(L_{a^{-1}} \circ L_{b^{-1}} \circ R_a)_b u, d(L_{a^{-1}} \circ L_{b^{-1}} \circ R_a)_b v \rangle \\
&= \langle d(R_a \circ L_{a^{-1}} \circ L_{b^{-1}})_b u, d(R_a \circ L_{a^{-1}} \circ L_{b^{-1}})_b v \rangle \\
&= \langle d(R_a \circ L_{a^{-1}})_e \circ d(L_{b^{-1}})_b u, d(R_a \circ L_{a^{-1}})_e \circ d(L_{b^{-1}})_b v \rangle \\
&= \langle \mathrm{Ad}_{a^{-1}} \circ d(L_{b^{-1}})_b u, \mathrm{Ad}_{a^{-1}} \circ d(L_{b^{-1}})_b v \rangle \\
&= \langle u, v \rangle,
\end{aligned}$$

as $\langle -, - \rangle$ is left-invariant and Ag_g-invariant for all $g \in G$. $\square$

Proposition 18.3 shows that if a Lie group, G, possesses a bi-invariant metric, then every linear map, Ad_a, is an orthogonal transformation of $\mathfrak{g}$. It follows that $\mathrm{Ad}(G)$ is a subgroup of the orthogonal group of $\mathfrak{g}$ and so, its closure, $\overline{\mathrm{Ad}(G)}$, is compact. It turns out that this condition is also sufficient!

To prove the above fact, we make use of an "averaging trick" used in representation theory. Recall that a *representation* of a Lie group, G, is a (smooth) homomorphism, $\rho\colon G \to \mathrm{GL}(V)$, where V is some finite-dimensional vector space. For any $g \in G$ and any $u \in V$, we often write $g \cdot u$ for $\rho(g)(u)$. We say that an inner-product, $\langle -, - \rangle$, on V is *G-invariant* iff

$$\langle g \cdot u, g \cdot v \rangle = \langle u, v \rangle,$$

for all $g \in G$ and all $u, v \in V$. If G is compact, then the "averaging trick", also called "Weyl's unitarian trick", yields the following important result:

Theorem 18.4 *If G is a compact Lie group, then for every representation, $\rho\colon G \to \mathrm{GL}(V)$, there is a G-invariant inner product on V.*

Proof. Recall from Section 9.4 that as a Lie group is orientable, it has a left-invariant volume form, ω, and for every continuous function, f, with compact support, we can define the integral, $\int_M f = \int_G f\omega$. Furthermore, when G is compact, we may assume that our integral is normalized so that $\int_G \omega = 1$ and in this case, our integral is both left and right invariant. Now, given any inner product, $\langle -, - \rangle$ on V, set

$$\langle\langle u, v \rangle\rangle = \int_G \langle g \cdot u, g \cdot v \rangle,$$

for all $u, v \in V$, where $\langle g \cdot u, g \cdot v \rangle$ denotes the function $g \mapsto \langle g \cdot u, g \cdot v \rangle$. It is easily checked that $\langle\langle -, - \rangle\rangle$ is an inner product on V. Furthermore, using the right-invariance of our integral (that is, $\int_G f = \int_G (R_h \circ f)$, for all $h \in G$), we have

$$\begin{aligned}
\langle\langle h \cdot u, h \cdot v \rangle\rangle &= \int_G \langle g \cdot (h \cdot u), g \cdot (h \cdot v) \rangle \\
&= \int_G \langle (gh) \cdot u, (gh) \cdot v \rangle \\
&= \int_G \langle g \cdot u, g \cdot v \rangle \\
&= \langle\langle u, v \rangle\rangle,
\end{aligned}$$

which shows that $\langle\langle -, - \rangle\rangle$ is G-invariant. □

Using Theorem 18.4, we can prove the following result giving a criterion for the existence of a G-invariant inner product for any representation of a Lie group, G (see Sternberg [143], Chapter 5, Theorem 5.2).

Theorem 18.5 *Let $\rho\colon G \to \mathrm{GL}(V)$ be a (finite-dimensional) representation of a Lie group, G. There is a G-invariant inner product on V iff $\overline{\rho(G)}$ is compact. In particular, if G is compact, then there is a G-invariant inner product on V.*

Proof. If V has a G-invariant inner product on V, then each linear map, $\rho(g)$, is an isometry, so $\rho(G)$ is a subgroup of the orthogonal group, $\mathbf{O}(V)$, of V. As $\mathbf{O}(V)$ is compact, $\overline{\rho(G)}$ is also compact.

Conversely, assume that $\overline{\rho(G)}$ is compact. In this case, $H = \overline{\rho(G)}$ is a closed subgroup of the lie group, $\mathrm{GL}(V)$, so by Theorem 5.12, H is a compact Lie subgroup of $\mathrm{GL}(V)$. Now, the inclusion homomorphism, $H \hookrightarrow \mathrm{GL}(V)$, is a representation of H ($f \cdot u = f(u)$, for all $f \in H$ and all $u \in V$), so by Theorem 18.4, there is an inner product on V which is H-invariant. However, for any $g \in G$, if we write $f = \rho(g) \in H$, then we have

$$\langle g \cdot u, g \cdot v\rangle = \langle f(u), f(v)\rangle = \langle u, v\rangle,$$

proving that $\langle -, -\rangle$ is G-invariant as well. □

Applying Theorem 18.5 to the adjoint representation, $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$, we get our second criterion for the existence of a bi-invariant metric on a Lie group.

Proposition 18.6 *Given any Lie group, G, an inner product, $\langle -, -\rangle$, on $\mathfrak{g}$ induces a bi-invariant metric on G iff $\overline{\mathrm{Ad}(G)}$ is compact. In particular, every compact Lie group has a bi-invariant metric.*

Proof. Proposition 18.3 is equivalent to the fact that G possesses a bi-invariant metric iff there is some Ad-invariant inner product on $\mathfrak{g}$. By Theorem 18.5, there is some Ad-invariant inner product on $\mathfrak{g}$ iff $\overline{\mathrm{Ad}(G)}$ is compact, which is the statement of our theorem. □

Proposition 18.6 can be used to prove that certain Lie groups do not have a bi-invariant metric. For example, Arsigny, Pennec and Ayache use Proposition 18.6 to give a short and elegant proof of the fact that $\mathbf{SE}(n)$ does not have any bi-invariant metric for all $n \geq 1$. As noted by these authors, other proofs found in the literature are a lot more complicated and only cover the case $n = 3$.

Recall the adjoint representation of $\mathfrak{g}$,

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),$$

given by $\mathrm{ad} = d\mathrm{Ad}_1$. Here is our third criterion for the existence of a bi-invariant metric on a connected Lie group.

Proposition 18.7 *If G is a connected Lie group, an inner product, $\langle -, -\rangle$, on $\mathfrak{g}$ induces a bi-invariant metric on G iff the linear map, $\mathrm{ad}(u)\colon \mathfrak{g} \to \mathfrak{g}$, is skew-adjoint for all $u \in \mathfrak{g}$, which means that*

$$\langle \mathrm{ad}(u)(v), w\rangle = -\langle v, \mathrm{ad}(u)(w)\rangle$$

for all $u, v, w \in \mathfrak{g}$ iff

$$\langle [x, y], z\rangle = \langle x, [y, z]\rangle$$

for all $x, y, z \in \mathfrak{g}$.

Proof. We follow Milnor [109], Lemma 7.2. By Proposition 18.3, an inner product on $\mathfrak{g}$ induces a bi-invariant metric on G iff Ad_g is an isometry for all $g \in G$. We know that we can choose a small enough open subset, U, of $\mathfrak{g}$ containing 0 so that $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism from U to $\exp(U)$. For any $g \in \exp(U)$, there is a unique, $u \in \mathfrak{g}$, so that $g = \exp(u)$. By Proposition 5.6,

$$\mathrm{Ad}(g) = \mathrm{Ad}(\exp(u)) = e^{\mathrm{ad}(u)}.$$

Now, $\mathrm{Ad}(g)$ is an isometry iff $\mathrm{Ad}(g)^{-1} = \mathrm{Ad}(g)^*$, where $\mathrm{Ad}(g)^*$ denotes the adjoint of $\mathrm{Ad}(g)$ and we know that

$$\mathrm{Ad}(g)^{-1} = e^{-\mathrm{ad}(u)} \quad \text{and} \quad \mathrm{Ad}(g)^* = e^{\mathrm{ad}(u)^*},$$

so we deduce that $\mathrm{Ad}(g)^{-1} = \mathrm{Ad}(g)^*$ iff

$$\mathrm{ad}(u)^* = -\mathrm{ad}(u),$$

which means that $\mathrm{ad}(u)$ is skew-adjoint. Since a connected Lie group is generated by any open subset containing the identity and since products of isometries are isometries, our results holds for all $g \in G$.

The skew-adjointness of $\mathrm{ad}(u)$ means that

$$\langle \mathrm{ad}(u)(v), w\rangle = -\langle v, \mathrm{ad}(u)(w)\rangle$$

for all $u, v, w \in \mathfrak{g}$ and since $\mathrm{ad}(u)(v) = [u, v]$ and $[u, v] = -[v, u]$, we get

$$\langle [v, u], w\rangle = \langle v, [u, w]\rangle$$

which is the last claim of the proposition after renaming u, v, w as y, x, z. $\square$

It will be convenient to say that an inner product on $\mathfrak{g}$ is *bi-invariant* iff every $\mathrm{ad}(u)$ is skew-adjoint.

If G is a connected Lie group, then the existence of a bi-invariant metric on G places a heavy restriction on its group structure as shown by the following result from Milnor's paper [109] (Lemma 7.5):

Theorem 18.8 *A connected Lie group, G, admits a bi-invariant metric iff it is isomorphic to the cartesian product of a compact group and a vector space ($\mathbb{R}^m$, for some $m \geq 0$).*

A proof of Theorem 18.8 can be found in Milnor [109] (Lemma 7.4 and Lemma 7.5). The proof uses the universal covering group and it is a bit involved. We will outline the structure of the proof because it is really quite beautiful.

In a first step, it is shown that if G has a bi-invariant metric, then its Lie algebra, $\mathfrak{g}$, can be written as an orthogonal coproduct

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k,$$

where each $\mathfrak{g}_i$ is either a simple ideal or a one-dimensional abelian ideal, that is, $\mathfrak{g}_i \cong \mathbb{R}$.

First, a few definitions.

Definition 18.2 A subset, $\mathfrak{h}$, of a Lie algebra, $\mathfrak{g}$, is a *Lie subalgebra* iff it is a subspace of $\mathfrak{g}$ (as a vector space) and if it is closed under the bracket operation on $\mathfrak{g}$. A subalgebra, $\mathfrak{h}$, is *abelian* iff $[x, y] = 0$ for all $x, y \in \mathfrak{h}$. An *ideal* in $\mathfrak{g}$ is a Lie subalgebra, $\mathfrak{h}$, such that

$$[h, g] \in \mathfrak{h}, \qquad \text{for all } h \in \mathfrak{h} \text{ and all } g \in \mathfrak{g}.$$

The *center*, $Z(\mathfrak{g})$, of a Lie algebra, $\mathfrak{g}$, is the set of all elements, $u \in \mathfrak{g}$, so that $[u, v] = 0$ for all $v \in \mathfrak{g}$, or equivalently, so that $\mathrm{ad}(u) = 0$. A Lie algebra, $\mathfrak{g}$, is *simple* iff it is non-abelian and if it has no ideal other than (0) and $\mathfrak{g}$. A Lie algebra, $\mathfrak{g}$, is *semisimple* iff it has no abelian ideal other than (0). A Lie group is *simple* (resp. *semisimple*) iff its Lie algebra is simple (resp. *semisimple*)

Clearly, the trivial subalgebras (0) and $\mathfrak{g}$ itself are ideals and the center is an abelian ideal.

Note that, by definition, simple and semisimple Lie algebras are non-abelian and a simple algebra is a semisimple algebra. It turns out that a Lie algebra, $\mathfrak{g}$, is semisimple iff it can be expressed as a direct sum of ideals, $\mathfrak{g}_i$, with each $\mathfrak{g}_i$ a simple algebra (see Knapp [89], Chapter I, Theorem 1.54). If we drop the requirement that a simple Lie algebra be non-abelian, thereby allowing one dimensional Lie algebras to be simple, we run into the trouble that a simple Lie algebra is no longer semisimple and the above theorem fails for this stupid reason. Thus, it seems technically advantageous to require that simple Lie algebras be non-abelian.

Nevertheless, in certain situations, it is desirable to drop the requirement that a simple Lie algebra be non-abelian and this is what Milnor does in his paper because it is more convenient for one of his proofs. This is a minor point but it could be confusing for uninitiated readers.

The next step is to lift the ideals, $\mathfrak{g}_i$, to the simply connected normal subgroups, G_i, of the universal covering group, $\widetilde{G}$, of $\mathfrak{g}$. For every simple ideal, $\mathfrak{g}_i$, in the decomposition it is proved that there is some constant, $c_i > 0$, so that all Ricci curvatures are strictly positive and bounded from below by c_i. Therefore, by Myers' Theorem (Theorem 13.28), G_i is compact. It follows that $\widetilde{G}$ is isomorphic to a product of compact simple Lie groups and some vector space, $\mathbb{R}^m$. Finally, we know that G is isomorphic to the quotient of $\widetilde{G}$ by a discrete normal subgroup of $\widetilde{G}$, which yields our theorem.

Because it is a fun proof, we prove the statement about the structure of a Lie algebra for which each $\mathrm{ad}(u)$ is skew-adjoint.

Proposition 18.9 *Let $\mathfrak{g}$ be a Lie algebra with an inner product such that the linear map, $\mathrm{ad}(u)$, is skew-adjoint for every $u \in \mathfrak{g}$. The orthogonal complement, $\mathfrak{a}^{\perp}$, of any ideal, $\mathfrak{a}$, is itself an ideal. Consequently, $\mathfrak{g}$ can be expressed as an orthogonal direct sum*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k,$$

where each $\mathfrak{g}_i$ is either a simple ideal or a one-dimensional abelian ideal, that is, $\mathfrak{g}_i \cong \mathbb{R}$.

Proof. Assume $u \in \mathfrak{g}$ is orthogonal to $\mathfrak{a}$. We need to prove that $[u, v]$ is orthogonal to $\mathfrak{a}$ for all $v \in \mathfrak{g}$. But, as $\mathrm{ad}(u)$ is skew-adjoint, $\mathrm{ad}(u)(v) = [u, v]$, and $\mathfrak{a}$ is an ideal, we have

$$\langle [u, v], a\rangle = -\langle u, [v, a]\rangle = 0, \qquad \text{for all } a \in \mathfrak{a},$$

which shows that $\mathfrak{a}^\perp$ is an ideal.

For the second statement, we use induction on the dimension of $\mathfrak{g}$ *but for this proof, we redefine a simple Lie algebra to be an algebra with no nontrivial proper ideals*. The case where $\dim \mathfrak{g} = 1$ is clear.

For the induction step, if $\mathfrak{g}$ is simple, we are done. Else, $\mathfrak{g}$ has some nontrivial proper ideal, $\mathfrak{h}$, and if we pick $\mathfrak{h}$ of minimal dimension, p, with $1 \leq p < n = \dim \mathfrak{g}$, then $\mathfrak{h}$ is simple. Now, $\mathfrak{h}^\perp$ is also an ideal and $\dim \mathfrak{h}^\perp < n$, so the induction hypothesis applies. Therefore, we have an orthogonal direct sum

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k,$$

where each $\mathfrak{g}_i$ is simple *in our relaxed sense*. However, if $\mathfrak{g}_i$ is not abelian, then it is simple in the usual sense and if $\mathfrak{g}_i$ is abelian, having no proper nontrivial ideal, it must be one-dimensional and we get our decomposition. □

We now investigate connections and curvature on Lie groups with a left-invariant metric.

18.3 Connections and Curvature of Left-Invariant Metrics on Lie Groups

If G is a Lie group equipped with a left-invariant metric, then it is possible to express the Levi-Civita connection and the sectional curvature in terms of quantities defined over the Lie algebra of G, at least for left-invariant vector fields. When the metric is bi-invariant, much nicer formulae can be obtained.

If $\langle -, -\rangle$ is a left-invariant metric on G, then for any two left-invariant vector fields, X, Y, we have

$$\langle X, Y\rangle_g = \langle X(g), Y(g)\rangle_g = \langle (dL_g)_e X(e), (dL_g)_e Y(e)\rangle_e = \langle X_e, Y_e\rangle_e = \langle X, Y\rangle_e,$$

which shows that the function, $g \mapsto \langle X, Y\rangle_g$, is constant. Therefore, for any vector field, Z,

$$Z(\langle X, Y\rangle) = 0.$$

If we go back to the Koszul formula (Proposition 11.18)

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X(\langle Y, Z\rangle) + Y(\langle X, Z\rangle) - Z(\langle X, Y\rangle) \\ &\quad - \langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle - \langle Z, [Y, X]\rangle, \end{aligned}$$

we deduce that for all left-invariant vector fields, X, Y, Z, we have

$$2\langle \nabla_X Y, Z\rangle = -\langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle - \langle Z, [Y, X]\rangle,$$

which can be rewritten as

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle. \tag{$\dagger$}$$

The above yields the formula

$$\nabla_u v = \frac{1}{2}\left([u, v] - \mathrm{ad}(u)^* v - \mathrm{ad}(v)^* u\right), \qquad u, v \in \mathfrak{g},$$

where $\mathrm{ad}(x)^*$ denotes the adjoint of $\mathrm{ad}(x)$.

Following Milnor, if we pick an orthonormal basis, $(e_1, \ldots, e_n)$, *w.r.t.* our inner product on $\mathfrak{g}$ and if we define the constants, α_{ijk}, by

$$\alpha_{ijk} = \langle [e_i, e_j], e_k\rangle,$$

we see that

$$\nabla_{e_i} e_j = \frac{1}{2}\sum_k (\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}) e_k. \tag{$*$}$$

Now, for orthonormal vectors, u, v, the sectional curvature is given by

$$K(u, v) = \langle R(u, v)u, v\rangle,$$

with

$$R(u, v) = \nabla_{[u,v]} - \nabla_u \nabla_v + \nabla_v \nabla_u.$$

If we plug the expressions from equation $(*)$ into the defintions we obtain the following proposition from Milnor [109] (Lemma 1.1):

Proposition 18.10 *Given a Lie group, G, equipped with a left-invariant metric, for any orthonormal basis, $(e_1, \ldots, e_n)$, of $\mathfrak{g}$ and with the structure constants, $\alpha_{ijk} = \langle [e_i, e_j], e_k\rangle$, the sectional curvature, $K(e_1, e_2)$, is given by*

$$\begin{aligned} K(e_1, e_2) &= \sum_k \frac{1}{2}(\alpha_{12k}(-\alpha_{12k} + \alpha_{2k1} + \alpha_{k12}) \\ &\quad -\frac{1}{4}(\alpha_{12k} - \alpha_{2k1} + \alpha_{k12})(\alpha_{12k} + \alpha_{2k1} - \alpha_{k12}) - \alpha_{k11}\alpha_{k22}). \end{aligned}$$

Although the above formula is not too useful in general, in some cases of interest, a great deal of cancellation takes place so that a more useful formula can be obtained. An example of this situation is provided by the next proposition (Milnor [109], Lemma 1.2).

Proposition 18.11 *Given a Lie group, G, equipped with a left-invariant metric, for any $u \in \mathfrak{g}$, if the linear map,* $\mathrm{ad}(u)$*, is self-adjoint then*

$$K(u, v) \geq 0$$

for all $v \in \mathfrak{g}$, where equality holds iff u is orthogonal to $[v, \mathfrak{g}] = \{[v, x] \mid x \in \mathfrak{g}\}$.

Proof. We may asssume that u and v are orthonormal. If we pick an orthonormal basis such that $e_1 = u$ and $e_2 = v$, the fact that $\mathrm{ad}(e_1)$ is skew-adjoint means that the array (α_{1jk}) is skew-symmetric (in the indices j and k). It follows that the formula of Proposition 18.10 reduces to

$$K(e_1, e_2) = \frac{1}{4} \sum_k \alpha_{2k1}^2,$$

so $K(e_1, e_2) \geq 0$, as claimed. Furthermore, $K(e_1, e_2) = 0$ iff $\alpha_{2k1} = 0$ for $k = 1, \ldots, n$, that is $\langle [e_2, e_k], e_1 \rangle = 0$ for $k = 1, \ldots, n$, which means that e_1 is orthogonal to $[e_2, \mathfrak{g}]$. □

Proposition 18.12 *Given a Lie group, G, equipped with a left-invariant metric, for any u in the center, $Z(\mathfrak{g})$, of $\mathfrak{g}$,*

$$K(u, v) \geq 0$$

for all $v \in \mathfrak{g}$.

Proof. For any element, u, in the center of $\mathfrak{g}$, we have $\mathrm{ad}(u) = 0$, and the zero map is obviously skew-adjoint. □

Recall that the Ricci curvature, $\mathrm{Ric}(u, v)$, is the trace of the linear map, $y \mapsto R(u, y)v$. With respect to any orthonormal basis, $(e_1, \ldots, e_n)$, of $\mathfrak{g}$, we have

$$\mathrm{Ric}(u, v) = \sum_{j=1}^{n} \langle R(u, e_j)v, e_j \rangle = \sum_{j=1}^{n} R(u, e_j, v, e_j).$$

The Ricci curvature is a symmetric form, so it is completely determined by the quadratic form

$$r(u) = \mathrm{Ric}(u, u) = \sum_{j=1}^{n} R(u, e_j, u, e_j).$$

When u is a unit vector, $r(u)$ is called the *Ricci curvature in the direction* u. If we pick an orthonormal basis such that $e_1 = u$, then

$$r(e_1) = \sum_{i=2}^{n} K(e_1, e_i).$$

For computational purposes it may be more convenient to introduce the *Ricci transformation*, $\widehat{r}$, defined by

$$\widehat{r}(x) = \sum_{i=1}^{n} R(e_i, x)e_i.$$

The Ricci transformation is self-adjoint and it is also the unique map so that

$$r(x) = \langle \widehat{r}(x), x \rangle, \qquad \text{for all } x \in \mathfrak{g}.$$

The eigenvalues of $\widehat{r}$ are called the *principal Ricci curvatures*.

Proposition 18.13 *Given a Lie group, G, equipped with a left-invariant metric, if the linear map, $\mathrm{ad}(u)$, is skew-adjoint, then $r(u) \geq 0$, where equality holds iff u is orthogonal to the commutator ideal, $[\mathfrak{g}, \mathfrak{g}]$.*

Proof. This follows from Proposition 18.11. □

In particular, if u is in the center of $\mathfrak{g}$, then $r(u) \geq 0$.

As a corollary of Proposition 18.13, we have the following result which is used in the proof of Theorem 18.8:

Proposition 18.14 *If G is a connected Lie group equipped with a bi-invariant metric and if the Lie algebra of G is simple, then there is a constant, $c > 0$, so that $r(u) \geq c$ for all unit vector, $u \in T_g G$, for all $g \in G$.*

Proof. First of all, the linear maps, $\mathrm{ad}(u)$, are skew-adjoint for all $u \in \mathfrak{g}$, which implies that $r(u) \geq 0$. As $\mathfrak{g}$ is simple, the commutator ideal, $[\mathfrak{g}, \mathfrak{g}]$ is either (0) or $\mathfrak{g}$. But, if $[\mathfrak{g}, \mathfrak{g}] = (0)$, then then $\mathfrak{g}$ is abelian, which is impossible since $\mathfrak{g}$ is simple. Therefore $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, which implies $r(u) > 0$ for all $u \neq 0$ (otherwise, u would be orthogonal to $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, which is impossible). As the set of unit vectors in $\mathfrak{g}$ is compact, the function, $u \mapsto r(u)$, achieves it minimum, c, and $c > 0$ as $r(u) > 0$ for all $u \neq 0$. But, $dL_g \colon \mathfrak{g} \to T_g G$ is an isometry for all $g \in G$, so $r(u) \geq c$ for all unit vectors, $u \in T_g G$, for all $g \in G$. □

By Myers' Theorem (Theorem 13.28), the Lie group G is compact and has a finite fundamental group.

The following interesting theorem is proved in Milnor (Milnor [109], Theorem 2.2):

Theorem 18.15 *A connected Lie group, G, admits a left-invariant metric with $r(u) > 0$ for all unit vectors $u \in \mathfrak{g}$ (all Ricci curvatures are strictly positive) iff G is compact and has finite fundamental group.*

The following criterion for obtaining a direction of negative curvature is also proved in Milnor (Milnor [109], Lemma 2.3):

Proposition 18.16 *Given a Lie group, G, equipped with a left-invariant metric, if u is orthogonal to the commutator ideal, $[\mathfrak{g}, \mathfrak{g}]$, then $r(u) \leq 0$, where equality holds iff $\mathrm{ad}(u)$ is self-adjoint.*

When G possesses a bi-invariant metric, much nicer formulae are obtained. First of all, as

$$\langle [u,v], w\rangle = \langle u, [v,w]\rangle,$$

the last two terms in equation (†) cancel out and we get

$$\nabla_u v = \frac{1}{2}\,[u,v],$$

for all $u, v \in \mathfrak{g}$. Then, we get

$$R(u,v) = \frac{1}{2}\mathrm{ad}([u,v]) - \frac{1}{4}\mathrm{ad}(u)\mathrm{ad}(v) + \frac{1}{4}\mathrm{ad}(v)\mathrm{ad}(u).$$

Using the Jacobi identity,

$$\mathrm{ad}([u,v]) = \mathrm{ad}(u)\mathrm{ad}(v) - \mathrm{ad}(v)\mathrm{ad}(u),$$

we get

$$R(u,v) = \frac{1}{4}\,\mathrm{ad}[u,v],$$

so

$$R(u,v)w = \frac{1}{4}\,[[u,v],w].$$

Hence, for unit orthogonal vectors, u, v, the sectional curvature, $K(u,v) = \langle R(u,v)u, v\rangle$, is given by

$$K(u,v) = \frac{1}{4}\,\langle [[u,v],u], v\rangle,$$

which (as $\langle [x,y], z\rangle = \langle x, [y,z]\rangle$) is rewritten as

$$K(u,v) = \frac{1}{4}\,\langle [u,v], [u,v]\rangle.$$

To compute the Ricci curvature, $\mathrm{Ric}(u,v)$, we observe that $\mathrm{Ric}(u,v)$ is the trace of the linear map,

$$y \mapsto R(u,y)v = \frac{1}{4}\,[[u,y],v] = -\frac{1}{4}\,[v,[u,y]] = -\frac{1}{4}\,\mathrm{ad}(v)\circ\mathrm{ad}(u)(y).$$

However, the bilinear form, B, on $\mathfrak{g}$, given by

$$B(u,v) = \mathrm{tr}(\mathrm{ad}(u)\circ\mathrm{ad}(v))$$

is a famous object known as the *Killing form* of the Lie algebra $\mathfrak{g}$. We will take a closer look at the Killing form shortly. For the time being, we observe that as $\mathrm{tr}(\mathrm{ad}(u)\circ\mathrm{ad}(v)) = \mathrm{tr}(\mathrm{ad}(v)\circ\mathrm{ad}(u))$, we get

$$\mathrm{Ric}(u,v) = -\frac{1}{4}\,B(u,v),$$

for all $u, v \in \mathfrak{g}$.

We summarize all this in

Proposition 18.17 *For any Lie group, G, equipped with a left-invariant metric, the following properties hold:*

(a) The connection, $\nabla_u v$, is given by

$$\nabla_u v = \frac{1}{2}\,[u, v], \qquad \textit{for all } u, v \in \mathfrak{g}$$

(b) The curvature tensor, $R(u, v)$, is given by

$$R(u, v) = \frac{1}{4}\,\mathrm{ad}[u, v], \qquad \textit{for all } u, v \in \mathfrak{g},$$

or equivalently,

$$R(u, v)w = \frac{1}{4}\,[[u, v], w], \qquad \textit{for all } u, v, w \in \mathfrak{g}.$$

(c) The sectional curvature, $K(u, v)$, is given by

$$K(u, v) = \frac{1}{4}\,\langle [u, v], [u, v]\rangle,$$

for all pairs of orthonormal vectors, $u, v \in \mathfrak{g}$.

(d) The Ricci curvature, $\mathrm{Ric}(u, v)$, is given by

$$\mathrm{Ric}(u, v) = -\frac{1}{4}\,B(u, v), \qquad \textit{for all } u, v \in \mathfrak{g},$$

where B is the Killing form, with

$$B(u, v) = \mathrm{tr}(\mathrm{ad}(u) \circ \mathrm{ad}(v)), \qquad \textit{for all } u, v \in \mathfrak{g}.$$

Consequently, $K(u, v) \geq 0$, with equality iff $[u, v] = 0$ and $r(u) \geq 0$, with equality iff u belongs to the center of $\mathfrak{g}$.

Remark: Proposition 18.17 shows that if a Lie group admits a bi-invariant metric, then its Killing form is negative semi-definite.

What are the geodesics in a Lie group equipped with a bi-invariant metric? The answer is simple: they are the integral curves of left-invariant vector fields.

Proposition 18.18 *For any Lie group, G, equipped with a bi-invariant metric, we have:*

(1) The inversion map, $\iota\colon g \mapsto g^{-1}$, is an isometry.

(2) For every $a \in G$, if I_a denotes the map given by

$$I_a(b) = ab^{-1}a, \qquad \text{for all } a, b \in G,$$

then I_a is an isometry fixing a which reverses geodesics, that is, for every geodesic, γ, through a

$$I_a(\gamma)(t) = \gamma(-t).$$

(3) The geodesics through e are the integral curves, $t \mapsto \exp(tu)$, where $u \in \mathfrak{g}$, that is, the one-parameter groups. Consequently, the Lie group exponential map, $\exp\colon \mathfrak{g} \to G$, coincides with the Riemannian exponential map (at e) from T_eG to G, where G is viewed as a Riemannian manifold.

Proof. (1) Since

$$\iota(g) = g^{-1} = g^{-1}h^{-1}h = (hg)^{-1}h = (R_h \circ \iota \circ L_h)(g),$$

we have

$$\iota = R_h \circ \iota \circ L_h, \qquad \text{for all } h \in G.$$

In particular, for $h = g^{-1}$, we get

$$d\iota_g = (dR_{g^{-1}})_e \circ d\iota_e \circ (dL_{g^{-1}})_g.$$

As $(dR_{g^{-1}})_e$ and $d(L_{g^{-1}})_g$ are isometries (since G has a bi-invariant metric), $d\iota_g$ is an isometry iff $d\iota_e$ is. Thus, it remains to show that $d\iota_e$ is an isometry. However, $d\iota_e = -\mathrm{id}$, so $d\iota_g$ is an isometry for all $g \in G$.

It remains to prove that $d\iota_e = -\mathrm{id}$. This can be done in several ways. If we denote the multiplication of the group by $\mu\colon G \times G \to G$, then $T_e(G \times G) = T_eG \oplus T_eG = \mathfrak{g} \oplus \mathfrak{g}$ and it is easy to see that

$$d\mu_{(e,e)}(u, v) = u + v, \qquad \text{for all } u, v \in \mathfrak{g}.$$

This is because $d\mu_{(e,e)}$ is a homomorphism and because $g \mapsto \mu(e, g)$ and $g \mapsto \mu(g, e)$ are the identity map. As the map, $g \mapsto \mu(g\iota(g))$, is the constant map with value e, by differentiating and using the chain rule, we get

$$d\iota_e(u) = -u,$$

as desired. (Another proof makes use of the fact that for every, $u \in \mathfrak{g}$, the integral curve, γ, through e with $\gamma'(0) = u$ is a group homomorphism. Therefore,

$$\iota(\gamma(t)) = \gamma(t)^{-1} = \gamma(-t)$$

and by differentiating, we get $d\iota_e(u) = -u$.)

(2) We follow Milnor [106] (Lemma 21.1). From (1), the map ι is an isometry so, by Proposition 13.8 (3), it preserves geodesics through e. Since $d\iota_e$ reverses $T_eG = \mathfrak{g}$, it reverses geodesics through e. Observe that

$$I_a = R_a \circ \iota \circ R_{a^{-1}},$$

so by (1), I_a is an isometry and obviously, $I_a(a) = a$. Again, by Proposition 13.8 (3), the isometry I_a preserve geodesics, and since R_a and $R_{a^{-1}}$ translate geodesics but ι reverses geodesics, it follows that I_a reverses geodesics.

(3) We follow Milnor [106] (Lemma 21.2). Assume γ is the unique geodesic through e such that $\gamma'(0) = u$, and let X be the left invariant vector field such that $X(e) = u$. The first step is to prove that γ has domain $\mathbb{R}$ and that it is a group homomorphism, that is,

$$\gamma(s+t) = \gamma(s)\gamma(t).$$

Details of this argument are given in Milnor [106] (Lemma 20.1 and Lemma 21.2) and in Gallot, Hulin and Lafontaine [60] (Appendix B, Solution of Exercise 2.90). We present Milnor's proof.

Claim. The isometries, I_a, have the following property: For every geodesic, γ, through a, if we let $p = \gamma(0)$ and $q = \gamma(r)$, then

$$I_q \circ I_p(\gamma(t)) = \gamma(t+2r),$$

whenever $\gamma(t)$ and $\gamma(t+2r)$ are defined.

Let $\alpha(t) = \gamma(t+r)$. Then, α is a geodesic with $\alpha(0) = q$. As I_p reverses geodesics through p (and similarly for I_q), we get

$$\begin{aligned} I_q \circ I_p(\gamma(t)) &= I_q(\gamma(-t)) \\ &= I_q(\alpha(-t-r)) \\ &= \alpha(t+r) = \gamma(t+2r). \end{aligned}$$

It follows from the claim that γ can be indefinitely extended, that is, the domain of γ is $\mathbb{R}$.

Next, we prove that γ is a homomorphism. By the Claim, $I_{\gamma(t)} \circ I_e$ takes $\gamma(u)$ into $\gamma(u+2t)$. Now, by definition of I_a and I_e,

$$I_{\gamma(t)} \circ I_e(a) = \gamma(t)a\gamma(t),$$

so, with $a = \gamma(u)$, we get

$$\gamma(t)\gamma(u)\gamma(t) = \gamma(u+2t).$$

By induction, it follows that

$$\gamma(nt) = \gamma(t)^n, \qquad \text{for all } n \in \mathbb{Z}.$$

We now use the (usual) trick of approximating every real by a rational number. For all $r, s \in \mathbb{R}$ with $s \neq 0$, if r/s is rational, say $r/s = m/n$ where m, n are integers, then $r = mt$ and $s = nt$ with $t = r/m = s/n$ and we get

$$\gamma(r+s) = \gamma(t)^{m+n} = \gamma(t)^m\gamma(t)^n = \gamma(r)\gamma(s).$$

Given any $t_1, t_2 \in \mathbb{R}$ with $t_2 \neq 0$, since t_1 and t_2 can be approximated by rationals r and s, as r/s is rational, $\gamma(r+s) = \gamma(r)\gamma(s)$, and by continuity, we get

$$\gamma(t_1 + t_2) = \gamma(t_1)\gamma(t_2),$$

as desired (the case $t_2 = 0$ is trivial as $\gamma(0) = e$).

As γ is a homomorphism, by differentiating the equation $\gamma(s+t) = \gamma(s)\gamma(t)$, we get

$$\frac{d}{dt}(\gamma(s+t))|_{t=0} = (dL_{\gamma(s)})_e \left(\frac{d}{dt}(\gamma(t))|_{t=0}\right),$$

that is

$$\gamma'(s) = (dL_{\gamma(s)})_e(\gamma'(0)) = X(\gamma(s)),$$

which means that γ is the integral curve of the left-invariant vector field, X, a one-parameter group.

Conversely, let c be the one-parameter group determined by a left-invariant vector field, X, with $X(e) = u$ and let γ be the unique geodesic through e such that $\gamma'(0) = u$. Since we have just shown that γ is a homomorphism with $\gamma'(0) = u$, by uniqueness of one-parameter groups, $c = \gamma$, that is, c is a geodesic. □

Remarks:

(1) As $R_g = \iota \circ L_{g^{-1}} \circ \iota$, we deduce that if G has a left-invariant metric, then this metric is also right-invariant iff ι is an isometry.

(2) Property (2) of Proposition 18.18 says that a Lie group with a bi-invariant metric is a *symmetric space*, an important class of Riemannian spaces invented and studied extensively by Elie Cartan.

(3) The proof of 18.18 (3) given in O'Neill [119] (Chapter 11, equivalence of (5) and (6) in Proposition 9) appears to be missing the "hard direction", namely, that a geodesic is a one-parameter group. Also, since left and right translations are isometries and since isometries map geodesics to geodesics, the geodesics through any point, $a \in G$, are the left (or right) translates of the geodesics through e and thus, are expressed in terms of the group exponential. Therefore, the geodesics through $a \in G$ are of the form

$$\gamma(t) = L_a(\exp(tu)),$$

where $u \in \mathfrak{g}$. Observe that $\gamma'(0) = (dL_a)_e(u)$.

(4) Some of the other facts stated in Proposition 18.17 and Proposition 18.18 are equivalent to the fact that a left-invariant metric is also bi-invariant, see O'Neill [119] (Chapter 11, Proposition 9).

Many more interesting results about left-invariant metrics on Lie groups can be found in Milnor's paper [109]. For example, flat left-invariant metrics on Lie a group are characterized (Theorem 1.5). We conclude this section by stating the following proposition (Milnor [109], Lemma 7.6):

Proposition 18.19 *If G is any compact, simple, Lie group, G, then the bi-invariant metric is unique up to a constant. Such a metric necessarily has constant Ricci curvature.*

18.4 The Killing Form

The Killing form showed the tip of its nose in Proposition 18.17. It is an important concept and, in this section, we establish some of its main properties. First, we recall its definition.

Definition 18.3 For any Lie algebra, $\mathfrak{g}$, the *Killing form, B, of* $\mathfrak{g}$ is the symmetric bilinear form, $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, given by

$$B(u, v) = \mathrm{tr}(\mathrm{ad}(u) \circ \mathrm{ad}(v)), \qquad \text{for all } u, v \in \mathfrak{g}.$$

If $\mathfrak{g}$ is the Lie algebra of a Lie group, G, we also refer to B as the *Killing form of G*.

Remark: According to the experts (see Knapp [89], page 754) the *Killing form* as above was not defined by Killing and is closer to a variant due to Elie Cartan. On the other hand, the notion of "Cartan matrix" is due to Wilhelm Killing!

For example, consider the group $\mathbf{SU}(2)$. Its Lie algebra, $\mathfrak{su}(2)$, consists of all skew-Hermitian 2×2 matrices with zero trace, that is matrices of the form

$$\begin{pmatrix} ai & b + ic \\ -b + ic & -ai \end{pmatrix}, \qquad a, b, c \in \mathbb{R},$$

a three-dimensional algebra. By picking a suitable basis of $\mathfrak{su}(2)$, it can be shown that

$$B(X, Y) = 4\mathrm{tr}(XY).$$

Now, if we consider the group $\mathbf{U}(2)$, its Lie algebra, $\mathfrak{u}(2)$, consists of all skew-Hermitian 2×2 matrices, that is matrices of the form

$$\begin{pmatrix} ai & b + ic \\ -b + ic & id \end{pmatrix}, \qquad a, b, c, d \in \mathbb{R},$$

a four-dimensional algebra. This time, it can be shown that

$$B(X, Y) = 4\mathrm{tr}(XY) - 2\mathrm{tr}(X)\mathrm{tr}(Y).$$

For $\mathbf{SO}(3)$, we know that $\mathfrak{so}(3) = \mathfrak{su}(2)$ and we get

$$B(X, Y) = \mathrm{tr}(XY).$$

Actually, it can be shown that

$$\begin{aligned} \mathbf{U}(n)\colon &\qquad B(X, Y) = 2n\mathrm{tr}(XY) - 2\mathrm{tr}(X)\mathrm{tr}(Y) \\ \mathbf{SU}(n)\colon &\qquad B(X, Y) = 2n\mathrm{tr}(XY) \\ \mathbf{SO}(n)\colon &\qquad B(X, Y) = (n-2)\mathrm{tr}(XY). \end{aligned}$$

Recall that a homomorphism of Lie algebras, $\varphi\colon \mathfrak{g} \to \mathfrak{h}$, is a linear map that preserves brackets, that is,

$$\varphi([u, v]) = [\varphi(u), \varphi(v)].$$

Proposition 18.20 *The Killing form, B, of a Lie algebra, $\mathfrak{g}$, has the following properties:*

(1) It is a symmetric bilinear form invariant under all automorphisms of $\mathfrak{g}$. In particular, if $\mathfrak{g}$ is the Lie algebra of a Lie group, G, then B is Ad_g-invariant, for all $g \in G$.

(2) The linear map, $\mathrm{ad}(u)$, is skew-adjoint w.r.t B for all $u \in \mathfrak{g}$, that is

$$B(\mathrm{ad}(u)(v), w) = -B(v, \mathrm{ad}(u)(w)), \qquad \text{for all } u, v, w \in \mathfrak{g}$$

or, equivalently

$$B([u, v], w) = B(u, [v, w]), \qquad \text{for all } u, v, w \in \mathfrak{g}.$$

Proof. (1) The form B is clearly bilinear and as $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, it is symmetric. If φ is an automorphism of $\mathfrak{g}$, the preservation of the bracket implies that

$$\mathrm{ad}(\varphi(u)) \circ \varphi = \varphi \circ \mathrm{ad}(u),$$

so

$$\mathrm{ad}(\varphi(u)) = \varphi \circ \mathrm{ad}(u) \circ \varphi^{-1}.$$

From $\mathrm{tr}(XY) = \mathrm{tr}(YX)$, we get $\mathrm{tr}(A) = \mathrm{tr}(BAB^{-1})$, so we get

$$\begin{aligned} B(\varphi(u), \varphi(v)) &= \mathrm{tr}(\mathrm{ad}(\varphi(u)) \circ \mathrm{ad}(\varphi(v)) \\ &= \mathrm{tr}(\varphi \circ \mathrm{ad}(u) \circ \varphi^{-1} \circ \varphi \circ \mathrm{ad}(v) \circ \varphi^{-1}) \\ &= \mathrm{tr}(\mathrm{ad}(u) \circ \mathrm{ad}(v)) = B(u, v). \end{aligned}$$

Since Ad_g is an automorphism of $\mathfrak{g}$ for all $g \in G$, B is Ad_g-invariant.

(2) We have

$$B(\mathrm{ad}(u)(v), w) = B([u, v], w) = \mathrm{tr}(\mathrm{ad}([u, v]) \circ \mathrm{ad}(w))$$

and

$$B(v, \mathrm{ad}(u)(w)) = B(v, [u, w]) = \mathrm{tr}(\mathrm{ad}(v) \circ \mathrm{ad}([u, w])).$$

However, the Jacobi identity is equivalent to

$$\mathrm{ad}([u, v]) = \mathrm{ad}(u) \circ \mathrm{ad}(v) - \mathrm{ad}(v) \circ \mathrm{ad}(u).$$

Consequently,

$$\begin{aligned} \mathrm{tr}(\mathrm{ad}([u, v]) \circ \mathrm{ad}(w)) &= \mathrm{tr}((\mathrm{ad}(u) \circ \mathrm{ad}(v) - \mathrm{ad}(v) \circ \mathrm{ad}(u)) \circ \mathrm{ad}(w)) \\ &= \mathrm{tr}(\mathrm{ad}(u) \circ \mathrm{ad}(v) \circ \mathrm{ad}(w)) - \mathrm{tr}(\mathrm{ad}(v) \circ \mathrm{ad}(u) \circ \mathrm{ad}(w)) \end{aligned}$$

and

$$\begin{aligned} \mathrm{tr}(\mathrm{ad}(v) \circ \mathrm{ad}([u, w])) &= \mathrm{tr}(\mathrm{ad}(v) \circ (\mathrm{ad}(u) \circ \mathrm{ad}(w) - \mathrm{ad}(w) \circ \mathrm{ad}(u))) \\ &= \mathrm{tr}(\mathrm{ad}(v) \circ \mathrm{ad}(u) \circ \mathrm{ad}(w)) - \mathrm{tr}(\mathrm{ad}(v) \circ \mathrm{ad}(w) \circ \mathrm{ad}(u)). \end{aligned}$$

As

$$\mathrm{tr}(\mathrm{ad}(u) \circ \mathrm{ad}(v) \circ \mathrm{ad}(w)) = \mathrm{tr}(\mathrm{ad}(v) \circ \mathrm{ad}(w) \circ \mathrm{ad}(u)),$$

we deduce that

$$B(\mathrm{ad}(u)(v), w) = \mathrm{tr}(\mathrm{ad}([u, v]) \circ \mathrm{ad}(w)) = -\mathrm{tr}(\mathrm{ad}(v) \circ \mathrm{ad}([u, w])) = -B(v, \mathrm{ad}(u)(w)),$$

as claimed. □

Remarkably, the Killing form yields a simple criterion due to Elie Cartan for testing whether a Lie algebra is semisimple.

Theorem 18.21 *(Cartan's Criterion for Semisimplicity) A lie algebra, $\mathfrak{g}$, is semisimple iff its Killing form, B, is non-degenerate.*

As far as we know, all the known proofs of Cartan's criterion are quite involved. A fairly easy going proof can be found in Knapp [89] (Chapter 1, Theorem 1.45). A more concise proof is given in Serre [136] (Chapter VI, Theorem 2.1). As a corollary of Theorem 18.21, we get:

Proposition 18.22 *If G is a semisimple Lie group, then the center of its Lie algebra is trivial, that is, $Z(\mathfrak{g}) = (0)$.*

Proof. Since $u \in \mathfrak{g}$ iff $\mathrm{ad}(u) = 0$, we have

$$B(u, u) = \mathrm{tr}(\mathrm{ad}(u) \circ \mathrm{ad}(u)) = 0.$$

As B is nondegenerate, we must have $u = 0$. □

Since a Lie group with trivial Lie algebra is discrete, this implies that the center of a simple Lie group is discrete (because the Lie algebra of the center of a Lie group is the center of its Lie algebra. Prove it!).

We can also characterize which Lie groups have a Killing form which is negative definite.

Theorem 18.23 *A connected Lie group is compact and semisimple iff its Killing form is negative definite.*

Proof. First, assume that G is compact and semisimple. Then, by Proposition 18.6, there is an inner product on $\mathfrak{g}$ inducing a bi-invariant metric on G and by Proposition 18.7, every linear map, $\mathrm{ad}(u)$, is skew-adjoint. Therefore, if we pick an orthonormal basis of $\mathfrak{g}$, the matrix, X, representing $\mathrm{ad}(u)$ is skew-symmetric and

$$B(u,u) = \mathrm{tr}(\mathrm{ad}(u)\circ \mathrm{ad}(u)) = \mathrm{tr}(XX) = \sum_{i,j=1}^{n} a_{ij}a_{ji} = -\sum_{i,j=1}^{n} a_{ij}^2 \leq 0.$$

Since G is semisimple, B is nondegenerate, and so, it is negative definite.

Now, assume that B is negative definite. If so, $-B$ is an inner product on $\mathfrak{g}$ and by Proposition 18.20, it is Ad-invariant. By Proposition 18.3, the inner product $-B$ induces a bi-invariant metric on G and by Proposition 18.17 (d), the Ricci curvature is given by

$$\mathrm{Ric}(u,v) = -\frac{1}{4}\,B(u,v),$$

which shows that $r(u) > 0$ for all units vectors, $u \in \mathfrak{g}$. As in the proof of Proposition 18.14, there is some constant, $c > 0$, which is a lower bound on all Ricci curvatures, $r(u)$, and by Myers' Theorem (Theorem 13.28), G is compact (with finite fundamental group). By Cartan's Criterion, as B is non-degenerate, G is also semisimple. □

Remark: A compact semisimple Lie group equipped with $-B$ as a metric is an Einstein manifold, since Ric is proportional to the metric (see Definition 13.5).

Using Theorem 18.23 and since the Killing forms for $\mathbf{U}(n)$, $\mathbf{SU}(n)$ and $\mathbf{S})(n)$ are given by

$$\begin{aligned} \mathbf{U}(n)\colon &\qquad B(X,Y) = 2n\mathrm{tr}(XY) - 2\mathrm{tr}(X)\mathrm{tr}(Y) \\ \mathbf{SU}(n)\colon &\qquad B(X,Y) = 2n\mathrm{tr}(XY) \\ \mathbf{SO}(n)\colon &\qquad B(X,Y) = (n-2)\mathrm{tr}(XY), \end{aligned}$$

we see that $\mathbf{SU}(n)$ and $\mathbf{SO}(n)$ are compact and semisimple but $\mathbf{U}(n)$, even though it is compact, is not semisimple.

Semisimple Lie algebras and semisimple Lie groups have been investigated extensively, starting with the complete classification of the complex semisimple Lie algebras by Killing (1888) and corrected by Elie Cartan in his thesis (1894). One should read the Notes, especially on Chapter II, at the end of Knapp's book [89] for a fascinating account of the history of the theory of semisimple Lie algebras.

The theories and the body of results that emerged from these investigations play a very important role not only in mathematics but also in physics and constitute one of the most beautiful chapters of mathematics. A quick introduction to these theories can be found in Arvanitoyeogos [8] and in Carter, Segal, Macdonald [31]. A more comprehensive but yet still introductory presentation is given in Hall [70]. The most comprehensive treatment is probably Knapp [89]. An older is classic is Helgason [73], which also discusses differential geometric aspects of Lie groups. Other "advanced" presentations can be found in Bröcker and tom Dieck [25], Serre [137, 136], Samelson [131], Humphreys [81] and Kirillov [86].

Chapter 19

The Log-Euclidean Framework Applied to SPD Matrices and Polyaffine Transformations

19.1 Introduction

In this Chapter, we use what we have learned in previous chapters to describe an approach due to Arsigny, Fillard, Pennec and Ayache to define a Lie group structure and a class of metrics on symmetric, positive-definite matrices (SPD matrices) which yield a new notion of mean on SPD matrices generalizing the standard notion of geometric mean.

SPD matrices are used in diffusion tensor magnetic resonance imaging (for short, DTI) and they are also a basic tool in numerical analysis, for example, in the generation of meshes to solve partial differential equations more efficiently.

As a consequence, there is a growing need to interpolate or to perform statistics on SPD matrices, such as computing the mean of a finite number of SPD matrices.

Recall that the set of $n \times n$ SPD matrices, $\mathbf{SPD}(n)$, is not a vector space (because if $A \in \mathbf{SPD}(n)$, then $\lambda A \notin \mathbf{SPD}(n)$ if $\lambda < 0$) but it is a convex cone. Thus, the *arithmetic mean* of n SPD matrices, $S_1, \ldots, S_n$, can be defined as $(S_1 + \cdots + S_n)/n$, which is SPD. However, there are many situations, especially in DTI, where this mean is not adequate. There are essentially two problems:

(1) The arithmetic mean is not invariant under inversion, which means that if $S = (S_1 + \cdots + S_n)/n$, then in general, $S^{-1} \neq (S_1^{-1} + \cdots + S_n^{-1})/n$.

(2) The swelling effect: the determinant, $\det(S)$, of the mean, S, may be strictly larger than the original determinants, $\det(S_i)$. This effect is undesirable in DTI because it amounts to introducing more diffusion, which is physically unacceptable.

To circumvent these difficulties, various metrics on SPD matrices have been proposed. One class of metrics is the *affine-invariant metrics* (see Arsigny, Pennec and Ayache [6]). The swelling effect disappears and the new mean is invariant under inversion but computing this new mean has a high computational cost and, in general, there is no closed-form formula for this new kind of mean.

Arsigny, Fillard, Pennec and Ayache [5] have defined a new family of metrics on $\mathbf{SPD}(n)$ named *Log-Euclidean metrics* and have also defined a novel structure of Lie group on $\mathbf{SPD}(n)$ which yields a notion of mean that has the same advantages as the affine mean but is a lot cheaper to compute. Furthermore, this new mean, called *Log-Euclidean mean*, is given by a simple closed-form formula. We will refer to this approach as the *Log-Euclidean Framework*.

The key point behind the Log-Euclidean Framework is the fact that the exponential map, $\exp\colon \mathbf{S}(n) \to \mathbf{SPD}(n)$, is a bijection, where $\mathbf{S}(n)$ is the space of $n \times n$ symmetric matrices (see Gallier [58], Chapter 14, Lemma 14.3.1). Consequently, the exponential map has a well-defined inverse, the *logarithm*, $\log\colon \mathbf{SPD}(n) \to \mathbf{S}(n)$.

But more is true. It turns out that $\exp\colon \mathbf{S}(n) \to \mathbf{SPD}(n)$ is a diffeomorphism, a fact stated as Theorem 2.8 in Arsigny, Fillard, Pennec and Ayache [5].

Since exp is a bijection, the above result follows from the fact that exp is a local diffeomorphism on $\mathbf{S}(n)$, because $d\exp_S$ is non-singular for all $S \in \mathbf{S}(n)$. In Arsigny, Fillard, Pennec and Ayache [5], it is proved that the non-singularity of $d\exp_I$ near 0, which is well-known, "propagates" to the whole of $\mathbf{S}(n)$.

Actually, the non-singularity of $d\exp$ on $\mathbf{S}(n)$ is a consequence of a more general result of some interest whose proof can be found in in Mmeimné and Testard [111], Chapter 3, Theorem 3.8.4 (see also Bourbaki [22], Chapter III, Section 6.9, Proposition 17, and also Theorem 6).

Let $\mathcal{S}(n)$ denote the set of all real matrices whose eigenvalues, $\lambda + i\mu$, lie in the horizontal strip determined by the condition $-\pi < \mu < \pi$. Then, we have the following theorem:

Theorem 19.1 *The restriction of the exponential map to $\mathcal{S}(n)$ is a diffeomorphism of $\mathcal{S}(n)$ onto its image, $\exp(\mathcal{S}(n))$. Furthermore, $\exp(\mathcal{S}(n))$ consists of all invertible matrices that have no real negative eigenvalues; it is an open subset of $\mathbf{GL}(n, \mathbb{R})$; it contains the open ball, $B(I, 1) = \{A \in \mathbf{GL}(n, \mathbb{R}) \mid \|A - I\| < 1\}$, for every norm $\|\ \|$ on $n \times n$ matrices satisfying the condition $\|AB\| \leq \|A\|\,\|B\|$.*

Part of the proof consists in showing that exp is a local diffeomorphism and for this, to prove that $d\exp_X$ is invertible for every $X \in \mathcal{S}(n)$. This requires finding an explicit formula for the derivative of the exponential, which can be done.

With this preparation we are ready to present the natural Lie group structure on $\mathbf{SPD}(n)$ introduced by Arsigny, Fillard, Pennec and Ayache [5] (see also Arsigny's thesis [3]).

19.2 A Lie-Group Structure on $\mathbf{SPD}(n)$

Using the diffeomorphism, $\exp\colon \mathbf{S}(n) \to \mathbf{SPD}(n)$, and its inverse, $\log\colon \mathbf{SPD}(n) \to \mathbf{S}(n)$, an abelian group structure can be defined on $\mathbf{SPD}(n)$ as follows:

Definition 19.1 For any two matrices, $S_1, S_2 \in \mathbf{SPD}(n)$, define the *logarithmic product*, $S_1 \odot S_2$, by

$$S_1 \odot S_2 = \exp(\log(S_1) + \log(S_2)).$$

Obviously, the multiplication operation, $\odot$, is commutative. The following proposition is shown in Arsigny, Fillard, Pennec and Ayache [5] (Proposition 3.2):

Proposition 19.2 *The set,* $\mathbf{SPD}(n)$*, with the binary operation,* $\odot$*, is an abelian group with identity,* I*, and with inverse operation the usual inverse of matrices. Whenever* S_1 *and* S_2 *commute, then* $S_1 \odot S_2 = S_1 S_2$ *(the usual multiplication of matrices).*

For the last statement, we need to show that if $S_1, S_2 \in \mathbf{SPD}(n)$ commute, then $S_1 S_2$ is also in $\mathbf{SPD}(n)$ and that $\log(S_1)$ and $\log(S_2)$ commute, which follows from the fact that if two diagonalizable matrices commute, then they can be diagonalized over the same basis of eigenvectors.

Actually, $(\mathbf{SPD}(n), \odot, I)$ is an abelian Lie group isomorphic to the vector space (also an abelian Lie group!) $\mathbf{S}(n)$, as shown in Arsigny, Fillard, Pennec and Ayache [5] (Theorem 3.3 and Proposition 3.4):

Theorem 19.3 *The abelian group,* $(\mathbf{SPD}(n), \odot, I)$ *is a Lie group isomorphic to its Lie algebra,* $\mathfrak{spd}(n) = \mathbf{S}(n)$*. In particular, the Lie group exponential in* $\mathbf{SPD}(n)$ *is identical to the usual exponential on* $\mathbf{S}(n)$*.*

We now investigate bi-invariant metrics on the Lie group, $\mathbf{SPD}(n)$.

19.3 Log-Euclidean Metrics on $\mathbf{SPD}(n)$

If G is a lie group, recall that we have the operations of left multiplication, L_a, and right multiplication, R_a, given by

$$L_a(b) = ab, \qquad R_a(b) = ba,$$

for all $a, b \in G$. A Riemannian metric, $\langle -, - \rangle$, on G is *left-invariant* iff dL_a is an isometry for all $a \in G$, that is,

$$\langle u, v \rangle_b = \langle (dL_a)_b(u), (dL_a)_b(v) \rangle_{ab},$$

for all $b \in G$ and all $u, v \in T_b G$. Similarly, $\langle -, - \rangle$ is *right-invariant* iff dR_a is an isometry for all $a \in G$ and $\langle -, - \rangle$ is *bi-invariant* iff it is both left and right invariant. In general, a Lie group does not admit a bi-invariant metric but an abelian Lie group always does because $\mathrm{Ad}_g = \mathrm{id} \in \mathbf{GL}(\mathfrak{g})$ for all $g \in G$ and so, the adjoint representation, $\mathrm{Ad}\colon G \to \mathbf{GL}(\mathfrak{g})$, is trivial (that is, $\mathrm{Ad}(G) = \{\mathrm{id}\}$) and then, the existence of bi-invariant metrics is a consequence of Proposition 18.3, which we repeat here for the convenience of the reader:

Proposition 19.4 *There is a bijective correspondence between bi-invariant metrics on a Lie group, G, and* Ad*-invariant inner products on the Lie algebra, $\mathfrak{g}$, of G, that is, inner products, $\langle -, - \rangle$, on $\mathfrak{g}$ such that Ad_a is an isometry of $\mathfrak{g}$ for all $a \in G$; more explicitly, inner products such that*

$$\langle \mathrm{Ad}_a u, \mathrm{Ad}_a v \rangle = \langle u, v \rangle,$$

for all $a \in G$ and all $u, v \in \mathfrak{g}$.

Then, given any inner product, $\langle -, - \rangle$ on G, the induced bi-invariant metric on G is given by

$$\langle u, v \rangle_g = \langle (dL_{g^{-1}})_g u, (dL_{g^{-1}})_g v \rangle.$$

Now, the geodesics on a Lie group equipped with a bi-invariant metric are the left (or right) translates of the geodesics through e and the geodesics through e are given by the group exponential, as stated in Proposition 18.18 (3) which we repeat for the convenience of the reader:

Proposition 19.5 *For any Lie group, G, equipped with a bi-invariant metric, we have:*

(1) The inversion map, $\iota\colon g \mapsto g^{-1}$, is an isometry.

(2) For every $a \in G$, if I_a denotes the map given by

$$I_a(b) = ab^{-1}a, \qquad \textit{for all } a, b \in G,$$

then I_a is an isometry fixing a which reverses geodesics, that is, for every geodesic, γ, through a

$$I_a(\gamma)(t) = \gamma(-t).$$

(3) The geodesics through e are the integral curves, $t \mapsto \exp(tu)$, where $u \in \mathfrak{g}$, that is, the one-parameter groups. Consequently, the Lie group exponential map, $\exp\colon \mathfrak{g} \to G$, coincides with the Riemannian exponential map (at e) from T_eG to G, where G is viewed as a Riemannian manifold.

If we apply Proposition 19.5 to the abelian Lie group, $\mathbf{SPD}(n)$, we find that the geodesics through S are of the form

$$\gamma(t) = S \odot e^{tV},$$

where $V \in \mathbf{S}(n)$. But $S = e^{\log S}$, so

$$S \odot e^{tV} = e^{\log S} \odot e^{tV} = e^{\log S + tV},$$

so every geodesic through S is of the form

$$\gamma(t) = e^{\log S + tV} = \exp(\log S + tV).$$

To avoid confusion between the exponential and the logarithm as Lie group maps and as Riemannian manifold maps, we will denote the former by exp and log and their Riemannian counterparts by Exp and Log. Note that

$$\gamma'(0) = d\exp_{\log S}(V)$$

and since the exponential map of $\mathbf{SPD}(n)$, as a Riemannian manifold, is given by

$$\mathrm{Exp}_S(U) = \gamma_U(1),$$

where γ_U is the unique geodesic such that $\gamma_U(0) = S$ and $\gamma'_U(0) = U$, we must have $d\exp_{\log S}(V) = U$, so $V = (d\exp_{\log S})^{-1}(U)$ and

$$\mathrm{Exp}_S(U) = e^{\log S + V} = e^{\log S + (d\exp_{\log S})^{-1}(U)}.$$

However, $\log \circ \exp = \mathrm{id}$ so, by differentiation, we get

$$(d\exp_{\log S})^{-1}(U) = d\log_S(U),$$

which yields

$$\mathrm{Exp}_S(U) = e^{\log S + d\log_S(U)}.$$

To get a formula for $\mathrm{Log}_S T$, we solve the equation $T = \mathrm{Exp}_S(U)$ with respect to U, that is

$$e^{\log S + (d\exp_{\log S})^{-1}(U)} = T$$

which yields

$$\log S + (d\exp_{\log S})^{-1}(U) = \log T,$$

that is, $U = d\exp_{\log S}(\log T - \log S)$. Therefore,

$$\mathrm{Log}_S T = d\exp_{\log S}(\log T - \log S).$$

Finally, we can find an explicit formula for the Riemannian metric,

$$\langle U, V\rangle_S = \langle d(L_{S^{-1}})_S(U), d(L_{S^{-1}})_S(V)\rangle,$$

because $d(L_{S^{-1}})_S = d\log_S$, which can be shown as follows: Observe that

$$(\log \circ L_{S^{-1}})(T) = \log S^{-1} + \log T,$$

so $d(\log \circ L_{S^{-1}})_T = d\log_T$, that is

$$d\log_{S^{-1}\odot T} \circ d(L_{S^{-1}})_T = d\log_T,$$

which, for $T = S$, yields $(dL_{S^{-1}})_S = d\log_S$, since $d\log_I = I$. Therefore,

$$\langle U, V\rangle_S = \langle d\log_S(U), d\log_S(V)\rangle.$$

Now, a Lie group with a bi-invariant metric is complete, so given any two matrices, $S, T \in \mathbf{SPD}(n)$, their distance is the length of the geodesic segment, γ_V, such that $\gamma_V(0) = S$ and $\gamma_V(1) = T$, namely $\|V\|$, but $V = \log_S T$ so that

$$d(S, T) = \|\log_S T\|_S\,,$$

where $\| \, \|_S$ is the norm given by the Riemannian metric. Using the equation

$$\mathrm{Log}_S T = d\exp_{\log S}(\log T - \log S),$$

and the fact that $d\log \circ d\exp = \mathrm{id}$, we get

$$d(S, T) = \|\log T - \log S\|\,,$$

where $\| \, \|$ is the norm corresponding to the inner product on $\mathfrak{spd}(n) = \mathbf{S}(n)$. Since $\langle -, -\rangle$ is a bi-invariant metric on $\mathbf{S}(n)$ and since

$$\langle U, V\rangle_S = \langle d\log_S(U), d\log_S(V)\rangle,$$

we see that the map, $\exp\colon \mathbf{S}(n) \to \mathbf{SPD}(n)$, is an isometry (since $d\exp \circ d\log = \mathrm{id}$).

In summary, we have proved Corollary 3.9 of Arsigny, Fillard, Pennec and Ayache [5]:

Theorem 19.6 *For any inner product, $\langle -, -\rangle$, on $\mathbf{S}(n)$, if we give the Lie group, $\mathbf{SPD}(n)$, the bi-invariant metric induced by $\langle -, -\rangle$, then the following properties hold:*

(1) For any $S \in \mathbf{SPD}(n)$, the geodesics through S are of the form

$$\gamma(t) = e^{\log S + tV}, \qquad V \in \mathbf{S}(n).$$

(2) The exponential and logarithm associated with the bi-invariant metric on $\mathbf{SPD}(n)$ are given by

$$\begin{aligned} \mathrm{Exp}_S(U) &= e^{\log S + d\log_S(U)} \\ \mathrm{Log}_S(T) &= d\exp_{\log S}(\log T - \log S), \end{aligned}$$

for all $S, T \in \mathbf{SPD}(n)$ and all $U \in \mathbf{S}(n)$.

(3) The bi-invariant metric on $\mathbf{SPD}(n)$ is given by

$$\langle U, V\rangle_S = \langle d\log_S(U), d\log_S(V)\rangle,$$

for all $U, V \in \mathbf{S}(n)$ and all $S \in \mathbf{SPD}(n)$ and the distance, $d(S, T)$, between any two matrices, $S, T \in \mathbf{SPD}(n)$, is given by

$$d(S, T) = \|\log T - \log S\|\,,$$

where $\| \, \|$ is the norm corresponding to the inner product on $\mathfrak{spd}(n) = \mathbf{S}(n)$.

(4) The map, $\exp\colon \mathbf{S}(n) \to \mathbf{SPD}(n)$*, is an isometry.*

In view of Theorem 19.6, part (3), bi-invariant metrics on the Lie group $\mathbf{SPD}(n)$ are called *Log-Euclidean metrics*. Since $\exp\colon \mathbf{S}(n) \to \mathbf{SPD}(n)$ is an isometry and $\mathbf{S}(n)$ is a vector space, the Riemannian Lie group, $\mathbf{SPD}(n)$, is a complete, simply-connected and flat manifold (the sectional curvature is zero at every point) that is, a flat *Hadamard manifold* (see Sakai [130], Chapter V, Section 4).

Although, in general, Log-Euclidean metrics are not invariant under the action of arbitary invertible matrices, they are invariant under similarity transformations (an isometry composed with a scaling). Recall that $\mathbf{GL}(n)$ acts on $\mathbf{SPD}(n)$, *via*,

$$A \cdot S = ASA^{\top},$$

for all $A \in \mathbf{GL}(n)$ and all $S \in \mathbf{SPD}(n)$. We say that a Log-Euclidean metric is *invariant under* $A \in \mathbf{GL}(n)$ iff

$$d(A \cdot S, A \cdot T) = d(S, T),$$

for all $S, T \in \mathbf{SPD}(n)$. The following result is proved in Arsigny, Fillard, Pennec and Ayache [5] (Proposition 3.11):

Proposition 19.7 *There exist metrics on* $\mathbf{S}(n)$ *that are invariant under all similarity transformations, for example, the metric* $\langle S, T\rangle = \mathrm{tr}(ST)$*.*

19.4 A Vector Space Structure on $\mathbf{SPD}(n)$

The vector space structure on $\mathbf{S}(n)$ can also be transfered onto $\mathbf{SPD}(n)$.

Definition 19.2 *For any matrix,* $S \in \mathbf{SPD}(n)$*, for any scalar,* $\lambda \in \mathbb{R}$*, define the scalar multiplication,* $\lambda \circledast S$*, by*

$$\lambda \circledast S = \exp(\lambda \log(S)).$$

It is easy to check that $(\mathbf{SPD}(n), \odot, \circledast)$ is a vector space with addition $\odot$ and scalar multiplication, $\circledast$. By construction, the map, $\exp\colon \mathbf{S}(n) \to \mathbf{SPD}(n)$, is a linear isomorphism. What happens is that the vector space structure on $\mathbf{S}(n)$ is transfered onto $\mathbf{SPD}(n)$ *via* the log and exp maps.

19.5 Log-Euclidean Means

One of the major advantages of Log-Euclidean metrics is that they yield a computationally inexpensive notion of mean with many desirable properties. If $(x_1, \ldots, x_n)$ is a list of n data points in $\mathbb{R}^m$, then it is an easy exercise to see that the mean, $\overline{x} = (x_1 + \cdots + x_n)/n$, is the unique minimum of the map

$$x \mapsto \sum_{i=1}^{n} d(x, x_i)_2^2,$$

where d_2 is the Euclidean distance on $\mathbb{R}^m$. We can think of the quantity,

$$\sum_{i=1}^{n} d(x, x_i)_2^2,$$

as the *dispersion* of the data. More generally, if (X, d) is a metric space, for any $\alpha > 0$ and any positive weights, $w_1, \ldots, w_n$, with $\sum_{i=1}^{n} w_i = 1$, we can consider the problem of minimizing the function,

$$x \mapsto \sum_{i=1}^{n} w_i d(x, x_i)^{\alpha}.$$

The case $\alpha = 2$ corresponds to a generalization of the notion of mean in a vector space and was investigated by Fréchet. In this case, any minimizer of the above function is known as a *Fréchet mean*. Fréchet means are not unique but if X is a complete Riemannian manifold, certain sufficient conditions on the dispersion of the data are known that ensure the existence and uniqueness of the Fréchet mean (see Pennec [120]). The case $\alpha = 1$ corresponds to a generalization of the notion of *median*. When the weights are all equal, the points that minimize the map,

$$x \mapsto \sum_{i=1}^{n} d(x, x_i),$$

are called *Steiner points*. On a Hadamard manifold, Steiner points can be characterized (see Sakai [130], Chapter V, Section 4, Proposition 4.9).

In the case where $X = \mathbf{SPD}(n)$ and d is a Log-Euclidean metric, it turns out that the Fréchet mean is unique and is given by a simple closed-form formula. This is easy to see and we have the following theorem from Arsigny, Fillard, Pennec and Ayache [5] (Theorem 3.13):

Theorem 19.8 *Given N matrices, $S_1, \ldots, S_N \in \mathbf{SPD}(n)$, their Log-Euclidean Fréchet mean exists and is uniquely determined by the formula*

$$\mathbb{E}_{\mathrm{LE}}(S_1, \ldots, S_N) = \exp\left(\frac{1}{N}\sum_{i=1}^{N} \log(S_i)\right).$$

Furthermore, the Log-Euclidean mean is similarity-invariant, invariant by group multiplication and inversion and exponential-invariant.

Similarity-invariance means that for any similarity, A,

$$\mathbb{E}_{\mathrm{LE}}(AS_1A^\top, \ldots, AS_NA^\top) = A\mathbb{E}_{\mathrm{LE}}(S_1, \ldots, S_N)A^\top$$

and similarly for the other types of invariance.

Observe that the Log-Euclidean mean is a generalization of the notion of geometric mean. Indeed, if $x_1, \ldots, x_n$ are n positive numbers, then their *geometric mean* is given by

$$\mathbb{E}_{\mathrm{geom}}(x_1, \ldots, x_n) = (x_1 \cdots x_n)^{\frac{1}{n}} = \exp\left(\frac{1}{n}\sum_{i=1}^{n} \log(x_i)\right).$$

The Log-Euclidean mean also has a good behavior with respect to determinants. The following theorem is proved in Arsigny, Fillard, Pennec and Ayache [5] (Theorem 4.2):

Theorem 19.9 *Given N matrices, $S_1, \ldots, S_N \in \mathbf{SPD}(n)$, we have*

$$\det(\mathbb{E}_{\mathrm{LE}}(S_1, \ldots, S_N)) = \mathbb{E}_{\mathrm{geom}}(\det(S_1), \ldots, \det(S_N)).$$

Remark: The last line of the proof in Arsigny, Fillard, Pennec and Ayache [5] seems incorrect.

Arsigny, Fillard, Pennec and Ayache [5] also compare the Log-Euclidean mean with the affine mean. We highly recommend the above paper as well as Arsigny's thesis [3] for further details.

Next, we discuss the application of the Log-Euclidean framework to the blending of locally affine transformations, known as Log-Euclidean polyaffine transformations, as presented in Arsigny, Commowick, Pennec and Ayache [4].

19.6 Log-Euclidean Polyaffine Transformations

The registration of medical images is an important and difficult problem. The work described in Arsigny, Commowick, Pennec and Ayache [4] (and Arsigny's thesis [3]) makes an orginal and valuable contribution to this problem by describing a method for parametrizing a class of non-rigid deformations with a small number of degrees of freedom. After a global affine alignment, this sort of parametrization allows a finer local registration with very smooth transformations. This type of parametrization is particularly well adpated to the registration of histological slices, see Arsigny, Pennec and Ayache [6].

The goal is to fuse some affine or rigid transformations in such a way that the resulting transformation is invertible and smooth. The direct approach which consists in blending N

global affine or rigid transformations, $T_1, \ldots, T_N$ using weights, $w_1, \ldots, w_N$, does not work because the resulting transformation,

$$T = \sum_{i=1}^{N} w_i T_i,$$

is not necessarily invertible. The purpose of the weights is to define the domain of influence in space of each T_i.

The key idea is to associate to each rigid (or affine) transformation, T, of $\mathbb{R}^n$, a vector field, V, and to view T as the diffeomorphism, Φ_1^V, corresponding to the time $t = 1$, where Φ_t^V is the global flow associated with V. In other words, T is the result of integrating an ODE

$$X' = V(X, t),$$

starting with some initial condition, X_0, and $T = X(1)$.

Now, it would be highly desirable if the vector field, V, did not depend on the time parameter, and this is indeed possible for a large class of affine transformations, which is one of the nice contributions of the work of Arsigny, Commowick, Pennec and Ayache [4]. Recall that an affine transformation, $X \mapsto LX + v$, (where L is an $n \times n$ matrix and $X, v \in \mathbb{R}^n$) can be conveniently represented as a linear transformation from $\mathbb{R}^{n+1}$ to itself if we write

$$\begin{pmatrix} X \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} L & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}.$$

Then, the ODE with constant coefficients

$$X' = LX + v,$$

can be written

$$\begin{pmatrix} X' \\ 0 \end{pmatrix} = \begin{pmatrix} L & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

and, for every initial condition, $X = X_0$, its unique solution is given by

$$\begin{pmatrix} X(t) \\ 1 \end{pmatrix} = \exp\left(t \begin{pmatrix} L & v \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} X_0 \\ 1 \end{pmatrix}.$$

Therefore, if we can find reasonable conditions on matrices, $T = \begin{pmatrix} M & t \\ 0 & 1 \end{pmatrix}$, to ensure that they have a unique real logarithm,

$$\log(T) = \begin{pmatrix} L & v \\ 0 & 0 \end{pmatrix},$$

then we will be able to associate a vector field, $V(X) = LX + v$, to T, in such a way that T is recovered by integrating the ODE, $X' = LX + v$. Furthermore, given N transformations,

$T_1, \ldots, T_N$, such that $\log(T_1), \ldots, \log(T_N)$ are uniquely defined, we can fuse $T_1, \ldots, T_N$ at the *infinitesimal level* by defining the ODE obtained by blending the vector fields, $V_1, \ldots, V_N$, associated with $T_1, \ldots, T_N$ (with $V_i(X) = L_iX + v_i$), namely

$$V(X) = \sum_{i=1}^{N} w_i(X)(L_iX + v_i).$$

Then, it is easy to see that the ODE,

$$X' = V(X),$$

has a unique solution for every $X = X_0$ defined for all t, and the fused transformation is just $T = X(1)$. Thus, the fused vector field,

$$V(X) = \sum_{i=1}^{N} w_i(X)(L_iX + v_i),$$

yields a one-parameter group of diffeomorphisms, Φ_t. Each transformation, Φ_t, is smooth and invertible and is called a *Log-Euclidean polyaffine tranformation*, for short, *LEPT*. Of course, we have the equation

$$\Phi_{s+t} = \Phi_s \circ \Phi_t,$$

for all $s, t \in \mathbb{R}$ so, in particular, the inverse of Φ_t is Φ_{-t}. We can also interpret Φ_s as $(\Phi_1)^s$, which will yield a fast method for computing Φ_s. Observe that when the weight are scalars, the one-parameter group is given by

$$\begin{pmatrix} \Phi_t(X) \\ 1 \end{pmatrix} = \exp\left(t \sum_{i=1}^{N} w_i \begin{pmatrix} L_i & v_i \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} X \\ 1 \end{pmatrix},$$

which is the Log-Euclidean mean of the affine transformations, T_i's (w.r.t. the weights w_i).

Fortunately, there is a sufficient condition for a real matrix to have a unique real logarithm and this condition is not too restrictive in practice.

Recall that $\mathcal{S}(n)$ denotes the set of all real matrices whose eigenvalues, $\lambda + i\mu$, lie in the horizontal strip determined by the condition $-\pi < \mu < \pi$. We have the following version of Theorem 19.1:

Theorem 19.10 *The image,* $\exp(\mathcal{S}(n))$*, of* $\mathcal{S}(n)$ *by the exponential map is the set of real invertible matrices with no negative eigenvalues and* $\exp\colon \mathcal{S}(n) \to \exp(\mathcal{S}(n))$ *is a bijection.*

Theorem 19.10 is stated in Kenney and Laub [84] without proof. Instead, Kenney and Laub cite DePrima and Johnson [41] for a proof but this latter paper deals with complex matrices and does not contain a proof of our result either. The injectivity part of Theorem 19.10 can be found in Mmeimné and Testard [111], Chapter 3, Theorem 3.8.4.

In fact, $\exp\colon \mathcal{S}(n) \to \exp(\mathcal{S}(n))$ is a diffeomorphism, a result proved in Bourbaki [22], see Chapter III, Section 6.9, Proposition 17 and Theorem 6. Curious readers should read Gallier [59] for the full story.

For any matrix, $A \in \exp(\mathcal{S}(n))$, we refer to the unique matrix, $X \in \mathcal{S}(n)$, such that $e^X = A$, as the *principal logarithm* of A and we denote it as $\log A$.

Observe that if T is an affine transformation given in matrix form by

$$T = \begin{pmatrix} M & t \\ 0 & 1 \end{pmatrix},$$

since the eigenvalues of T are those of M plus the eigenvalue 1, the matrix T has no negative eigenvalues iff M has no negative eigenvalues and thus the principal logarithm of T exists iff the principal logarithm of M exists.

It is proved in Arsigny, Commowick, Pennec and Ayache that LEPT's are affine invariant, see [4], Section 2.3. This shows that LEPT's are produced by a truly geometric kind of blending, since the result does not depend at all on the choice of the coordinate system.

In the next section, we describe a fast method for computing due to Arsigny, Commowick, Pennec and Ayache [4].

19.7 Fast Polyaffine Transforms

Recall that since LEPT's are members of the one-parameter group, $(\Phi_t)_{t\in\mathbb{R}}$, we have

$$\Phi_{2t} = \Phi_{t+t} = \Phi_t^2$$

and thus,

$$\Phi_1 = (\Phi_{1/2^N})^{2^N}.$$

Observe the formal analogy of the above formula with the formula

$$\exp(M) = \exp\left(\frac{M}{2^N}\right)^{2^N},$$

for computing the exponential of a matrix, M, by the *scaling and squaring method*.

It turns out that the "scaling and squaring method" is one of the most efficient methods for computing the exponential of a matrix, see Kenney and Laub [84] and Higham [74]. The key idea is that $\exp(M)$ is easy to compute if M is close zero since, in this case, one can use a few terms of the exponential series, or better, a Padé approximant (see Higham [74]). The scaling and squaring method for computing the exponential of a matrix, M, can be sketched as follows:

1. *Scaling Step*: Divide M by a factor, 2^N, so that $\frac{M}{2^N}$ is close enough to zero.

2. *Exponentiation step*: Compute $\exp\left(\frac{M}{2^N}\right)$ with high precision, for example, using a Padé approximant.

3. *Squaring Step*: Square $\exp\left(\frac{M}{2^N}\right)$ repeatedly N times to obtain $\exp\left(\frac{M}{2^N}\right)^{2^N}$, a very accurate approximation of e^M.

There is also a so-called *inverse scaling and squaring method* to compute efficiently the principal logarithm of a real matrix, see Cheng, Higham, Kenney and Laub [32].

Arsigny, Commowick, Pennec and Ayache made the very astute observation that the scaling and squaring method can be adpated to compute LEPT's very efficiently [4]. This method, called *fast polyaffine transform*, computes the values of a Log-Euclidean polyaffine transformation, $T = \Phi_1$, at the vertices of a regular n-dimensional grid (in practice, for $n = 2$ or $n = 3$). Recall that T is obtained by integrating an ODE, $X' = V(X)$, where the vector field, V, is obtained by blending the vector fields associated with some affine transformations, $T_1, \ldots, T_n$, having a principal logarithm.

Here are the three steps of the **fast polyaffine transform**:

1. *Scaling Step*: Divide the vector field, V, by a factor, 2^N, so that $\frac{V}{2^N}$ is close enough to zero.

2. *Exponentiation step*: Compute $\Phi_{1/2^N}$, using some adequate numerical integration method.

3. *Squaring Step*: Compose $\Phi_{1/2^N}$ with itself recursively N times to obtain an accurate approximation of $T = \Phi_1$.

Of course, one has to provide practical methods to achieve step 2 and step 3. Several methods to achieve step 2 and step 3 are proposed in Arsigny, Commowick, Pennec and Ayache [4]. One also has to worry about boundary effects, but this problem can be alleviated too, using bounding boxes. At this point, the reader is urged to read the full paper [4] for complete details and beautiful pictures illustrating the use of LEPT's in medical imaging.

To conclude our survey of the Log-Euclidean polyaffine framework for locally affine registration, we briefly discuss how the Log-Euclidean framework can be generalized to rigid and affine transformations.

19.8 A Log-Euclidean Framework for Transformations in $\exp(\mathcal{S}(n))$

Arsigny, Commowick, Pennec and Ayache observed that if T_1 and T_2 are two affine transformations in $\exp(\mathcal{S}(n))$, then we can define their distance as

$$d(T_1, T_2) = \|\log(T_1) - \log(T_2)\|,$$

where $\| \ \|$ is any norm on $n \times n$ matrices (see [4], Appendix A.1). We can go a little further and make $\mathcal{S}(n)$ and $\exp(\mathcal{S}(n))$ into Riemannian manifolds in such a way that the exponential map, $\exp\colon \mathcal{S}(n) \to \exp(\mathcal{S}(n))$, is an isometry.

Since $\mathcal{S}(n)$ is an open subset of the vector space, $\mathrm{M}(n, \mathbb{R})$, of all $n \times n$ real matrices, $\mathcal{S}(n)$ is a manifold, and since $\exp(\mathcal{S}(n))$ is an open subset of the manifold, $\mathbf{GL}(n, \mathbb{R})$, it is also a manifold. Obviously, $T_L\mathcal{S}(n) \cong \mathrm{M}(n, \mathbb{R})$ and $T_S \exp(\mathcal{S}(n)) \cong \mathrm{M}(n, \mathbb{R})$, for all $L \in \mathcal{S}(n)$ and all $S \in \exp(\mathcal{S}(n))$ and the maps, $d\exp_L\colon T_L\mathcal{S}(n) \to T_{\exp(L)} \exp(\mathcal{S}(n))$ and $d\log_S\colon T_S \exp(\mathcal{S}(n)) \to T_{\log(S)}\mathcal{S}(n)$, are linear isomorphisms. We can make $\mathcal{S}(n)$ into a Riemannian manifold by giving it the induced metric induced by any norm, $\| \ \|$, on $\mathrm{M}(n, \mathbb{R})$, and make $\exp(\mathcal{S}(n))$ into a Riemannian manifold by defining the metric, $\langle -, - \rangle_S$, on $T_S \exp(\mathcal{S}(n))$, by

$$\langle A, B\rangle_S = \|d\log_S(A) - d\log_S(B)\|,$$

for all $S \in \exp(\mathcal{S}(n))$ and all $A, B \in \mathrm{M}(n, \mathbb{R})$. Then, it is easy to check that $\exp\colon \mathcal{S}(n) \to \exp(\mathcal{S}(n))$ is indeed an isometry and, as a consequence, the Riemannian distance between two matrices, $T_1, T_2 \in \exp(\mathcal{S}(n))$, is given by

$$d(T_1, T_2) = \|\log(T_1) - \log(T_2)\|,$$

again called the *Log-Euclidean distance*.

Since every affine transformation, T, can be represented in matrix form as

$$T = \begin{pmatrix} M & t \\ 0 & 1 \end{pmatrix},$$

and, as we saw in section 19.6, since the principal logarithm of T exists iff the principal logarithm of M exists, we can view the set of affine transformations that have a principal logarithm as a subset of $\exp(\mathcal{S}(n + 1))$.

Unfortunately, this time, even though they are both flat, $\mathcal{S}(n)$ and $\exp(\mathcal{S}(n))$ are not complete manifolds and so, the Fréchet mean of N matrices, $T_1, \ldots, T_n \in \exp(\mathcal{S}(n))$, may not exist.

However, recall that from Theorem 19.1 that the open ball,

$$B(I, 1) = \{A \in \mathbf{GL}(n, \mathbb{R}) \mid \|A - I\|' < 1\},$$

is contained in $\exp(\mathcal{S}(n))$ for any norm, $\| \ \|'$, on matrices (not necessarily equal to the norm defining the Riemannian metric on $\mathcal{S}(n)$) such that $\|AB\|' \leq \|A\|' \, \|B\|'$ so, for any matrices $T_1, \ldots, T_n \in B(I, 1)$, the Fréchet mean is well defined and is uniquely determined by

$$\mathbb{E}_{\mathrm{LE}}(T_1, \ldots, T_N) = \exp\left(\frac{1}{N}\sum_{i=1}^{N} \log(T_i)\right),$$

namely, it is their *Log-Euclidean mean*.

From a practical point of view, one only needs to ckeck that the eigenvalues, ξ, of $\frac{1}{N}\sum_{i=1}^{N}\log(T_i)$ are in the horizontal strip, $-\pi < \Im(\xi) < \pi$.

Provided that $\mathbb{E}_{\mathrm{LE}}(T_1, \ldots, T_N)$ is defined, it is easy to show, as in the case of SPD matrices, that $\det(\mathbb{E}_{\mathrm{LE}}(T_1, \ldots, T_N))$ is the geometric mean of the determinants of the T_i's.

The Riemannian distance on $\exp(\mathcal{S}(n))$ is not affine invariant but it is invariant under inversion, under rescaling by a positive scalar, and under rotation for certain norms on $\mathcal{S}(n)$ (see [4], Appendix A.2). However, the Log-Euclidean mean of matrices in $\exp(\mathcal{S}(n))$ is invariant under conjugation by any matrix, $A \in \mathbf{GL}(n, \mathbb{R})$, since $ASA^{-1} \in \exp(\mathcal{S}(n))$ for any $S \in \exp(\mathcal{S}(n))$ and since $\log(ASA^{-1}) = A\log(S)A^{-1}$. In particular, the Log-Euclidean mean of affine transformations in $\exp(\mathcal{S}(n+1))$ is invariant under arbitrary invertible affine transformations (again, see [4], Appendix A.2).

For more details on the Log-Euclidean framework for locally rigid or affine deformation, for example, about regularization, the reader should read Arsigny, Commowick, Pennec and Ayache [4].

Chapter 20

Fréchet Mean and Statistics on Riemannian Manifolds; Applications to Medical Image Analysis

Chapter 21

Clifford Algebras, Clifford Groups, and the Groups Pin(n) and Spin(n)

21.1 Introduction: Rotations As Group Actions

The main goal of this chapter is to explain how rotations in $\mathbb{R}^n$ are induced by the action of a certain group, $\mathbf{Spin}(n)$, on $\mathbb{R}^n$, in a way that generalizes the action of the unit complex numbers, $\mathbf{U}(1)$, on $\mathbb{R}^2$, and the action of the unit quaternions, $\mathbf{SU}(2)$, on $\mathbb{R}^3$ (*i.e.*, the action is defined in terms of multiplication in a larger algebra containing both the group $\mathbf{Spin}(n)$ and $\mathbb{R}^n$). The group $\mathbf{Spin}(n)$, called a *spinor group*, is defined as a certain subgroup of units of an algebra, Cl_n, the *Clifford algebra* associated with $\mathbb{R}^n$. Furthermore, for $n \geq 3$, we are lucky, because the group $\mathbf{Spin}(n)$ is topologically simpler than the group $\mathbf{SO}(n)$. Indeed, for $n \geq 3$, the group $\mathbf{Spin}(n)$ is simply connected (a fact that it not so easy to prove without some machinery), whereas $\mathbf{SO}(n)$ is not simply connected. Intuitively speaking, $\mathbf{SO}(n)$ is more twisted than $\mathbf{Spin}(n)$. In fact, we will see that $\mathbf{Spin}(n)$ is a double cover of $\mathbf{SO}(n)$.

Since the spinor groups are certain well chosen subroups of units of Clifford algebras, it is necessary to investigate Clifford algebras to get a firm understanding of spinor groups. This chapter provides a tutorial on Clifford algebra and the groups **Spin** and **Pin**, including a study of the structure of the Clifford algebra $\mathrm{Cl}_{p,q}$ associated with a nondegenerate symmetric bilinear form of signature (p, q) and culminating in the beautiful "8-periodicity theorem" of Elie Cartan and Raoul Bott (with proofs). We also explain when $\mathbf{Spin}(p, q)$ is a double-cover of $\mathbf{SO}(p, q)$. The reader should be warned that a certain amount of algebraic (and topological) background is expected. This being said, perseverant readers will be rewarded by being exposed to some beautiful and nontrivial concepts and results, including Elie Cartan and Raoul Bott "8-periodicity theorem."

Going back to rotations as transformations induced by group actions, recall that if V is a vector space, a *linear action (on the left) of a group G on V* is a map, $\alpha\colon G \times V \to V$, satisfying the following conditions, where, for simplicity of notation, we denote $\alpha(g, v)$ by $g \cdot v$:

(1) $g \cdot (h \cdot v) = (gh) \cdot v$, for all $g, h \in G$ and $v \in V$;

(2) $1 \cdot v = v$, for all $v \in V$, where 1 is the identity of the group G;

(3) The map $v \mapsto g \cdot v$ is a linear isomorphism of V for every $g \in G$.

For example, the (multiplicative) group, $\mathbf{U}(1)$, of unit complex numbers acts on $\mathbb{R}^2$ (by identifying $\mathbb{R}^2$ and $\mathbb{C}$) *via* complex multiplication: For every $z = a + ib$ (with $a^2 + b^2 = 1$), for every $(x, y) \in \mathbb{R}^2$ (viewing (x, y) as the complex number $x + iy$),

$$z \cdot (x, y) = (ax - by, ay + bx).$$

Now, every unit complex number is of the form $\cos\theta + i\sin\theta$, and thus, the above action of $z = \cos\theta + i\sin\theta$ on $\mathbb{R}^2$ corresponds to the rotation of angle θ around the origin. In the case $n = 2$, the groups $\mathbf{U}(1)$ and $\mathbf{SO}(2)$ are isomorphic, but this is an exception.

We can define an action of the group of unit quaternions, $\mathbf{SU}(2)$, on $\mathbb{R}^3$. For this, we use the fact that $\mathbb{R}^3$ can be identified with the pure quaternions in $\mathbb{H}$, namely, the quaternions of the form $x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, where $(x_1, x_2, x_3) \in \mathbb{R}^3$. Then, we define the action of $\mathbf{SU}(2)$ over $\mathbb{R}^3$ by

$$Z \cdot X = ZXZ^{-1} = ZX\overline{Z},$$

where $Z \in \mathbf{SU}(2)$ and X is any pure quaternion. Now, it turns out that the map ρ_Z (where $\rho_Z(X) = ZX\overline{Z}$) is indeed a rotation, and that the map $\rho\colon Z \mapsto \rho_Z$ is a surjective homomorphism, $\rho\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$, whose kernel is $\{-\mathbf{1}, \mathbf{1}\}$, where $\mathbf{1}$ denotes the multiplicative unit quaternion. (For details, see Gallier [58], Chapter 8).

We can also define an action of the group $\mathbf{SU}(2) \times \mathbf{SU}(2)$ over $\mathbb{R}^4$, by identifying $\mathbb{R}^4$ with the quaternions. In this case,

$$(Y, Z) \cdot X = YX\overline{Z},$$

where $(Y, Z) \in \mathbf{SU}(2) \times \mathbf{SU}(2)$ and $X \in \mathbb{H}$ is any quaternion. Then, the map $\rho_{Y,\overline{Z}}$ is a rotation (where $\rho_{Y,\overline{Z}}(X) = YX\overline{Z}$), and the map $\rho\colon (Y, Z) \mapsto \rho_{Y,\overline{Z}}$ is a surjective homomorphism, $\rho\colon \mathbf{SU}(2) \times \mathbf{SU}(2) \to \mathbf{SO}(4)$, whose kernel is $\{(\mathbf{1}, \mathbf{1}), (-\mathbf{1}, -\mathbf{1})\}$. (For details, see Gallier [58], Chapter 8).

Thus, we observe that for $n = 2, 3, 4$, the rotations in $\mathbf{SO}(n)$ can be realized *via* the linear action of some group (the case $n = 1$ is trivial, since $\mathbf{SO}(1) = \{1, -1\}$). It is also the case that the action of each group can be somehow be described in terms of multiplication in some larger algebra "containing" the original vector space $\mathbb{R}^n$ ($\mathbb{C}$ for $n = 2$, $\mathbb{H}$ for $n = 3, 4$). However, these groups appear to have been discovered in an ad hoc fashion, and there does not appear to be any universal way to define the action of these groups on $\mathbb{R}^n$. It would certainly be nice if the action was always of the form

$$Z \cdot X = ZXZ^{-1} (= ZX\overline{Z}).$$

A systematic way of constructing groups realizing rotations in terms of linear action, using a uniform notion of action, does exist. Such groups are the spinor groups, to be described in the following sections.

21.2 Clifford Algebras

We explained in Section 21.1 how the rotations in $\mathbf{SO}(3)$ can be realized by the linear action of the group of unit quaternions, $\mathbf{SU}(2)$, on $\mathbb{R}^3$, and how the rotations in $\mathbf{SO}(4)$ can be realized by the linear action of the group $\mathbf{SU}(2) \times \mathbf{SU}(2)$ on $\mathbb{R}^4$.

The main reasons why the rotations in $\mathbf{SO}(3)$ can be represented by unit quaternions are the following:

(1) For every nonzero vector $u \in \mathbb{R}^3$, the reflection s_u about the hyperplane perpendicular to u is represented by the map
$$v \mapsto -uvu^{-1},$$
where u and v are viewed as pure quaternions in $\mathbb{H}$ (i.e., if $u = (u_1, u_2, u_2)$, then view u as $u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, and similarly for v).

(2) The group $\mathbf{SO}(3)$ is generated by the reflections.

As one can imagine, a successful generalization of the quaternions, i.e., the discovery of a group, G inducing the rotations in $\mathbf{SO}(n)$ *via* a linear action, depends on the ability to generalize properties (1) and (2) above. Fortunately, it is true that the group $\mathbf{SO}(n)$ is generated by the hyperplane reflections. In fact, this is also true for the orthogonal group, $\mathbf{O}(n)$, and more generally, for the group of direct isometries, $\mathbf{O}(\Phi)$, of any nondegenerate quadratic form, Φ, by the *Cartan-Dieudonné theorem* (for instance, see Bourbaki [20], or Gallier [58], Chapter 7, Theorem 7.2.1). In order to generalize (2), we need to understand how the group G acts on $\mathbb{R}^n$. Now, the case $n = 3$ is special, because the underlying space, $\mathbb{R}^3$, on which the rotations act, can be embedded as the pure quaternions in $\mathbb{H}$. The case $n = 4$ is also special, because $\mathbb{R}^4$ is the underlying space of $\mathbb{H}$. The generalization to $n \geq 5$ requires more machinery, namely, the notions of Clifford groups and Clifford algebras. As we will see, for every $n \geq 2$, there is a compact, connected (and simply connected when $n \geq 3$) group, $\mathbf{Spin}(n)$, the "spinor group," and a surjective homomorphism, $\rho\colon \mathbf{Spin}(n) \to \mathbf{SO}(n)$, whose kernel is $\{-1, 1\}$. This time, $\mathbf{Spin}(n)$ acts directly on $\mathbb{R}^n$, because $\mathbf{Spin}(n)$ is a certain subgroup of the group of units of the *Clifford algebra*, Cl_n, and $\mathbb{R}^n$ is naturally a subspace of Cl_n.

The group of unit quaternions $\mathbf{SU}(2)$ turns out to be isomorphic to the spinor group $\mathbf{Spin}(3)$. Because $\mathbf{Spin}(3)$ acts directly on $\mathbb{R}^3$, the representation of rotations in $\mathbf{SO}(3)$ by elements of $\mathbf{Spin}(3)$ may be viewed as more natural than the representation by unit quaternions. The group $\mathbf{SU}(2) \times \mathbf{SU}(2)$ turns out to be isomorphic to the spinor group $\mathbf{Spin}(4)$, but this isomorphism is less obvious.

In summary, we are going to define a group $\mathbf{Spin}(n)$ representing the rotations in $\mathbf{SO}(n)$, for any $n \geq 1$, in the sense that there is a linear action of $\mathbf{Spin}(n)$ on $\mathbb{R}^n$ which induces a surjective homomorphism, $\rho\colon \mathbf{Spin}(n) \to \mathbf{SO}(n)$, whose kernel is $\{-1, 1\}$. Furthermore, the action of $\mathbf{Spin}(n)$ on $\mathbb{R}^n$ is given in terms of multiplication in an algebra, Cl_n, containing $\mathbf{Spin}(n)$, and in which $\mathbb{R}^n$ is also embedded. It turns out that as a bonus, for $n \geq 3$, the group $\mathbf{Spin}(n)$ is topologically simpler than $\mathbf{SO}(n)$, since $\mathbf{Spin}(n)$ is simply connected, but $\mathbf{SO}(n)$ is not. By being astute, we can also construct a group, $\mathbf{Pin}(n)$, and a linear action of $\mathbf{Pin}(n)$ on $\mathbb{R}^n$ that induces a surjective homomorphism, $\rho\colon \mathbf{Pin}(n) \to \mathbf{O}(n)$, whose kernel is $\{-1, 1\}$. The difficulty here is the presence of the negative sign in (2). We will see how Atiyah, Bott and Shapiro circumvent this problem by using a "twisted adjoint action," as opposed to the usual adjoint action (where $v \mapsto uvu^{-1}$).

Our presentation is heavily influenced by Bröcker and tom Dieck [25], Chapter 1, Section 6, where most details can be found. This Chapter is almost entirely taken from the first 11 pages of the beautiful and seminal paper by Atiyah, Bott and Shapiro [11], Clifford Modules, and we highly recommend it. Another excellent (but concise) exposition can be found in Kirillov [85]. A very thorough exposition can be found in two places:

1. Lawson and Michelsohn [96], where the material on $\mathbf{Pin}(p, q)$ and $\mathbf{Spin}(p, q)$ can be found in Chapter I.

2. Lounesto's excellent book [99].

One may also want to consult Baker [13], Curtis [38], Porteous [124], Fulton and Harris (Lecture 20) [57], Choquet-Bruhat [36], Bourbaki [20], or Chevalley [35], a classic. The original source is Elie Cartan's book (1937) whose translation in English appears in [28].

We begin by recalling what is an algebra over a field. Let K denote any (commutative) field, although for our purposes, we may assume that $K = \mathbb{R}$ (and occasionally, $K = \mathbb{C}$). Since we will only be dealing with associative algebras with a multiplicative unit, we only define algebras of this kind.

Definition 21.1 Given a field, K, a *K-algebra* is a K-vector space, A, together with a bilinear operation, $\cdot\colon A \times A \to A$, called *multiplication*, which makes A into a ring with unity, 1 (or 1_A, when we want to be very precise). This means that $\cdot$ is associative and that there is a multiplicative identity element, 1, so that $1 \cdot a = a \cdot 1 = a$, for all $a \in A$. Given two K-algebras A and B, a *K-algebra homomorphism, $h\colon A \to B$*, is a linear map that is also a ring homomorphism, with $h(1_A) = 1_B$.

For example, the ring, $M_n(K)$, of all $n \times n$ matrices over a field, K, is a K-algebra.

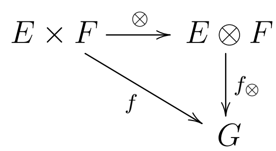
There is an obvious notion of *ideal* of a K-algebra: An ideal, $\mathfrak{A} \subseteq A$, is a linear subspace of A that is also a two-sided ideal with respect to multiplication in A. If the field K is understood, we usually simply say an algebra instead of a K-algebra.

We will also need tensor products. A rather detailed exposition of tensor products is given in Chapter 22 and the reader may want to review Section 22.1. For the reader's convenience, we recall the definition of the tensor product of vector spaces. The basic idea is that tensor products allow us to view multilinear maps as linear maps. The maps become simpler, but the spaces (product spaces) become more complicated (tensor products). For more details, see Section 22.1 or Atiyah and Macdonald [9].

Definition 21.2 Given two K-vector spaces, E and F, a *tensor product of* E *and* F is a pair, $(E \otimes F, \otimes)$, where $E \otimes F$ is a K-vector space and $\otimes \colon E \times F \to E \otimes F$ is a bilinear map, so that for every K-vector space, G, and every bilinear map, $f \colon E \times F \to G$, there is a unique linear map, $f_{\otimes} \colon E \otimes F \to G$, with

$$f(u, v) = f_{\otimes}(u \otimes v) \quad \text{for all } u \in E \text{ and all } v \in V,$$

as in the diagram below:



The vector space $E \otimes F$ is defined up to isomorphism. The vectors $u \otimes v$, where $u \in E$ and $v \in F$, generate $E \otimes F$.

Remark: We should really denote the tensor product of E and F by $E \otimes_K F$, since it depends on the field K. Since we usually deal with a fixed field K, we use the simpler notation $E \otimes F$.

As shown in Section 22.3, we have natural isomorphisms

$$(E \otimes F) \otimes G \approx E \otimes (F \otimes G) \quad \text{and} \quad E \otimes F \approx F \otimes E.$$

Given two linear maps $f \colon E \to F$ and $g \colon E' \to F'$, we have a unique bilinear map $f \times g \colon E \times E' \to F \times F'$ so that

$$(f \times g)(a, a') = (f(a), g(a')) \quad \text{for all } a \in E \text{ and all } a' \in E'.$$

Thus, we have the bilinear map $\otimes \circ (f \times g) \colon E \times E' \to F \otimes F'$, and so, there is a unique linear map $f \otimes g \colon E \otimes E' \to F \otimes F'$, so that

$$(f \otimes g)(a \otimes a') = f(a) \otimes g(a') \quad \text{for all } a \in E \text{ and all } a' \in E'.$$

Let us now assume that E and F are K-algebras. We want to make $E \otimes F$ into a K-algebra. Since the multiplication operations $m_E \colon E \times E \to E$ and $m_F \colon F \times F \to F$ are bilinear, we get linear maps $m'_E \colon E \otimes E \to E$ and $m'_F \colon F \otimes F \to F$, and thus, the linear map

$$m'_E \otimes m'_F \colon (E \otimes E) \otimes (F \otimes F) \to E \otimes F.$$

Using the isomorphism $\tau \colon (E \otimes E) \otimes (F \otimes F) \to (E \otimes F) \otimes (E \otimes F)$, we get a linear map

$$m_{E \otimes F} \colon (E \otimes F) \otimes (E \otimes F) \to E \otimes F,$$

which defines a multiplication m on $E \otimes F$ (namely, $m(u, v) = m_{E \otimes F}(u \otimes v)$). It is easily checked that $E \otimes F$ is indeed a K-algebra under the multiplication m. Using the simpler notation $\cdot$ for m, we have

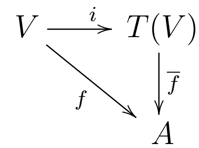
$$(a \otimes a') \cdot (b \otimes b') = (ab) \otimes (a'b')$$

for all $a, b \in E$ and all $a', b' \in F$.

Given any vector space, V, over a field, K, there is a special K-algebra, $T(V)$, together with a linear map, $i\colon V \to T(V)$, with the following universal mapping property: Given any K-algebra, A, for any linear map, $f\colon V \to A$, there is a unique K-algebra homomorphism, $\overline{f}\colon T(V) \to A$, so that

$$f = \overline{f} \circ i,$$

as in the diagram below:



The algebra, $T(V)$, is the *tensor algebra of* V, see Section 22.5. The algebra $T(V)$ may be constructed as the direct sum

$$T(V) = \bigoplus_{i \geq 0} V^{\otimes i},$$

where $V^0 = K$, and $V^{\otimes i}$ is the i-fold tensor product of V with itself. For every $i \geq 0$, there is a natural injection $\iota_n\colon V^{\otimes n} \to T(V)$, and in particular, an injection $\iota_0\colon K \to T(V)$. The multiplicative unit, $\mathbf{1}$, of $T(V)$ is the image, $\iota_0(1)$, in $T(V)$ of the unit, 1, of the field K. Since every $v \in T(V)$ can be expressed as a finite sum

$$v = v_1 + \cdots + v_k,$$

where $v_i \in V^{\otimes n_i}$ and the n_i are natural numbers with $n_i \neq n_j$ if $i \neq j$, to define multiplication in $T(V)$, using bilinearity, it is enough to define the multiplication $V^{\otimes m} \times V^{\otimes n} \longrightarrow V^{\otimes (m+n)}$. Of course, this is defined by

$$(v_1 \otimes \cdots \otimes v_m) \cdot (w_1 \otimes \cdots \otimes w_n) = v_1 \otimes \cdots \otimes v_m \otimes w_1 \otimes \cdots \otimes w_n.$$

(This has to be made rigorous by using isomorphisms involving the associativity of tensor products, for details, see see Atiyah and Macdonald [9].) The algebra $T(V)$ is an example of a *graded algebra*, where the *homogeneous elements of rank* n are the elements in $V^{\otimes n}$.

Remark: It is important to note that multiplication in $T(V)$ is **not** commutative. Also, in all rigor, the unit, $\mathbf{1}$, of $T(V)$ is **not equal** to 1, the unit of the field K. However, in view of the injection $\iota_0\colon K \to T(V)$, for the sake of notational simplicity, we will denote $\mathbf{1}$ by 1. More generally, in view of the injections $\iota_n\colon V^{\otimes n} \to T(V)$, we identify elements of $V^{\otimes n}$ with their images in $T(V)$.

Most algebras of interest arise as well-chosen quotients of the tensor algebra $T(V)$. This is true for the *exterior algebra,* $\bigwedge^{\bullet} V$ (also called *Grassmann algebra*), where we take the quotient of $T(V)$ modulo the ideal generated by all elements of the form $v \otimes v$, where $v \in V$, see Section 22.15.

A Clifford algebra may be viewed as a refinement of the exterior algebra, in which we take the quotient of $T(V)$ modulo the ideal generated by all elements of the form $v \otimes v - \Phi(v) \cdot 1$, where Φ is the quadratic form associated with a symmetric bilinear form, $\varphi\colon V \times V \to K$, and $\cdot\colon K \times T(V) \to T(V)$ denotes the scalar product of the algebra $T(V)$. For simplicity, let us assume that we are now dealing with real algebras.

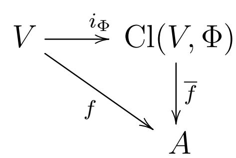
Definition 21.3 Let V be a real finite-dimensional vector space together with a symmetric bilinear form, $\varphi\colon V \times V \to \mathbb{R}$, and associated quadratic form, $\Phi(v) = \varphi(v, v)$. A *Clifford algebra associated with V and Φ* is a real algebra, $\mathrm{Cl}(V, \Phi)$, together with a linear map, $i_\Phi\colon V \to \mathrm{Cl}(V, \Phi)$, satisfying the condition $(i(v))^2 = \Phi(v) \cdot 1$ for all $v \in V$ and so that for every real algebra, A, and every linear map, $f\colon V \to A$, with

$$(f(v))^2 = \Phi(v) \cdot 1 \quad \text{for all } v \in V,$$

there is a unique algebra homomorphism, $\overline{f}\colon \mathrm{Cl}(V, \Phi) \to A$, so that

$$f = \overline{f} \circ i_\Phi,$$

as in the diagram below:



We use the notation, $\lambda \cdot u$, for the product of a scalar, $\lambda \in \mathbb{R}$, and of an element, u, in the algebra $\mathrm{Cl}(V, \Phi)$ and juxtaposition, uv, for the multiplication of two elements, u and v, in the algebra $\mathrm{Cl}(V, \Phi)$.

By a familiar argument, any two Clifford algebras associated with V and Φ are isomorphic. We often denote i_Φ by i.

To show the existence of $\mathrm{Cl}(V, \Phi)$, observe that $T(V)/\mathfrak{A}$ does the job, where $\mathfrak{A}$ is the ideal of $T(V)$ generated by all elements of the form $v \otimes v - \Phi(v) \cdot 1$, where $v \in V$. The map $i_\Phi\colon V \to \mathrm{Cl}(V, \Phi)$ is the composition

$$V \xrightarrow{\iota_1} T(V) \xrightarrow{\pi} T(V)/\mathfrak{A},$$

where π is the natural quotient map. We often denote the Clifford algebra $\mathrm{Cl}(V, \Phi)$ simply by $\mathrm{Cl}(\Phi)$.

Remark: Observe that Definition 21.3 does not assert that i_Φ is injective or that there is an injection of $\mathbb{R}$ into $\mathrm{Cl}(V, \Phi)$, but we will prove later that both facts are true when V is finite-dimensional. Also, as in the case of the tensor algebra, the unit of the algebra $\mathrm{Cl}(V, \Phi)$ and the unit of the field $\mathbb{R}$ are **not equal**.

Since

$$\Phi(u+v) - \Phi(u) - \Phi(v) = 2\varphi(u,v)$$

and

$$(i(u+v))^2 = (i(u))^2 + (i(v))^2 + i(u)i(v) + i(v)i(u),$$

using the fact that

$$i(u)^2 = \Phi(u)\cdot 1,$$

we get

$$i(u)i(v) + i(v)i(u) = 2\varphi(u,v)\cdot 1.$$

As a consequence, if $(u_1, \ldots, u_n)$ is an orthogonal basis w.r.t. φ (which means that $\varphi(u_j, u_k) = 0$ for all $j \neq k$), we have

$$i(u_j)i(u_k) + i(u_k)i(u_j) = 0 \quad \text{for all } j \neq k.$$

Remark: Certain authors drop the unit, 1, of the Clifford algebra $\mathrm{Cl}(V, \Phi)$ when writing the identities

$$i(u)^2 = \Phi(u)\cdot 1$$

and

$$2\varphi(u,v)\cdot 1 = i(u)i(v) + i(v)i(u),$$

where the second identity is often written as

$$\varphi(u,v) = \frac{1}{2}(i(u)i(v) + i(v)i(u)).$$

This is very confusing and technically wrong, because we only have an injection of $\mathbb{R}$ into $\mathrm{Cl}(V, \Phi)$, but $\mathbb{R}$ is **not** a subset of $\mathrm{Cl}(V, \Phi)$.

We warn the readers that Lawson and Michelsohn [96] adopt the opposite of our sign convention in defining Clifford algebras, i.e., they use the condition

$$(f(v))^2 = -\Phi(v)\cdot 1 \quad \text{for all } v \in V.$$

The most confusing consequence of this is that their $\mathrm{Cl}(p,q)$ is our $\mathrm{Cl}(q,p)$.

Observe that when $\Phi \equiv 0$ is the quadratic form identically zero everywhere, then the Clifford algebra $\mathrm{Cl}(V, 0)$ is just the exterior algebra, $\bigwedge^\bullet V$.

Example 21.1 Let $V = \mathbb{R}$, $e_1 = 1$, and assume that $\Phi(x_1e_1) = -x_1^2$. Then, $\mathrm{Cl}(\Phi)$ is spanned by the basis $(1, e_1)$. We have

$$e_1^2 = -1.$$

Under the bijection

$$e_1 \mapsto i,$$

the Clifford algebra, $\mathrm{Cl}(\Phi)$, also denoted by Cl_1, is isomorphic to the algebra of complex numbers, $\mathbb{C}$.

Now, let $V = \mathbb{R}^2$, (e_1, e_2) be the canonical basis, and assume that $\Phi(x_1e_1 + x_2e_2) = -(x_1^2 + x_2^2)$. Then, $\mathrm{Cl}(\Phi)$ is spanned by the basis by $(1, e_1, e_2, e_1e_2)$. Furthermore, we have

$$e_2e_1 = -e_1e_2, \quad e_1^2 = -1, \quad e_2^2 = -1, \quad (e_1e_2)^2 = -1.$$

Under the bijection

$$e_1 \mapsto \mathbf{i}, \quad e_2 \mapsto \mathbf{j}, \quad e_1e_2 \mapsto \mathbf{k},$$

it is easily checked that the quaternion identities

$$\begin{aligned} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 &= -\mathbf{1}, \\ \mathbf{ij} = -\mathbf{ji} &= \mathbf{k}, \\ \mathbf{jk} = -\mathbf{kj} &= \mathbf{i}, \\ \mathbf{ki} = -\mathbf{ik} &= \mathbf{j}, \end{aligned}$$

hold, and thus, the Clifford algebra $\mathrm{Cl}(\Phi)$, also denoted by Cl_2, is isomorphic to the algebra of quaternions, $\mathbb{H}$.

Our prime goal is to define an action of $\mathrm{Cl}(\Phi)$ on V in such a way that by restricting this action to some suitably chosen multiplicative subgroups of $\mathrm{Cl}(\Phi)$, we get surjective homomorphisms onto $\mathbf{O}(\Phi)$ and $\mathbf{SO}(\Phi)$, respectively. The key point is that a reflection in V about a hyperplane H orthogonal to a vector w can be defined by such an action, but some negative sign shows up. A correct handling of signs is a bit subtle and requires the introduction of a canonical anti-automorphism, t, and of a canonical automorphism, α, defined as follows:

Proposition 21.1 *Every Clifford algebra,* $\mathrm{Cl}(\Phi)$*, possesses a canonical anti-automorphism,* $t\colon \mathrm{Cl}(\Phi) \to \mathrm{Cl}(\Phi)$*, satisfying the properties*

$$t(xy) = t(y)t(x), \quad t \circ t = \mathrm{id}, \quad \textit{and} \quad t(i(v)) = i(v),$$

for all $x, y \in \mathrm{Cl}(\Phi)$ *and all* $v \in V$*. Furthermore, such an anti-automorphism is unique.*

Proof. Consider the opposite algebra $\mathrm{Cl}(\Phi)^o$, in which the product of x and y is given by yx. It has the universal mapping property. Thus, we get a unique isomorphism, t, as in the diagram below:

$$\begin{array}{ccc} V & \xrightarrow{i} & \mathrm{Cl}(V, \Phi) \\ & \searrow^{i} & \downarrow t \\ & & \mathrm{Cl}(\Phi)^o \end{array}$$

□

We also denote $t(x)$ by x^t. When V is finite-dimensional, for a more palatable description of t in terms of a basis of V, see the paragraph following Theorem 21.4.

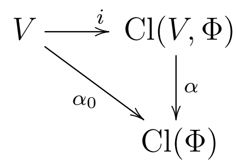
The canonical automorphism, α, is defined using the proposition

Proposition 21.2 *Every Clifford algebra,* $\mathrm{Cl}(\Phi)$*, has a unique canonical automorphism,* $\alpha\colon \mathrm{Cl}(\Phi) \to \mathrm{Cl}(\Phi)$*, satisfying the properties*

$$\alpha \circ \alpha = \mathrm{id}, \quad \textit{and} \quad \alpha(i(v)) = -i(v),$$

for all $v \in V$*.*

Proof. Consider the linear map $\alpha_0\colon V \to \mathrm{Cl}(\Phi)$ defined by $\alpha_0(v) = -i(v)$, for all $v \in V$. We get a unique homomorphism, α, as in the diagram below:



Furthermore, every $x \in \mathrm{Cl}(\Phi)$ can be written as

$$x = x_1 \cdots x_m,$$

with $x_j \in i(V)$, and since $\alpha(x_j) = -x_j$, we get $\alpha \circ \alpha = \mathrm{id}$. It is clear that α is bijective. $\square$

Again, when V is finite-dimensional, a more palatable description of α in terms of a basis of V can be given. If $(e_1, \ldots, e_n)$ is a basis of V, then the Clifford algebra $\mathrm{Cl}(\Phi)$ consists of certain kinds of "polynomials," linear combinations of monomials of the form $\sum_J \lambda_J e_J$, where $J = \{i_1, i_2, \ldots, i_k\}$ is any subset (possibly empty) of $\{1, \ldots, n\}$, with $1 \leq i_1 < i_2 \cdots < i_k \leq n$, and the monomial e_J is the "product" $e_{i_1} e_{i_2} \cdots e_{i_k}$. The map α is the linear map defined on monomials by

$$\alpha(e_{i_1} e_{i_2} \cdots e_{i_k}) = (-1)^k e_{i_1} e_{i_2} \cdots e_{i_k}.$$

For a more rigorous explanation, see the paragraph following Theorem 21.4.

We now show that if V has dimension n, then i is injective and $\mathrm{Cl}(\Phi)$ has dimension 2^n. A clever way of doing this is to introduce a graded tensor product.

First, observe that

$$\mathrm{Cl}(\Phi) = \mathrm{Cl}^0(\Phi) \oplus \mathrm{Cl}^1(\Phi),$$

where

$$\mathrm{Cl}^i(\Phi) = \{x \in \mathrm{Cl}(\Phi) \mid \alpha(x) = (-1)^i x\}, \quad \text{where } i = 0, 1.$$

We say that we have a $\mathbb{Z}/2$-*grading*, which means that if $x \in \mathrm{Cl}^i(\Phi)$ and $y \in \mathrm{Cl}^j(\Phi)$, then $xy \in \mathrm{Cl}^{i+j \pmod 2}(\Phi)$.

When V is finite-dimensional, since every element of $\mathrm{Cl}(\Phi)$ is a linear combination of the form $\sum_J \lambda_J e_J$, as explained earlier, in view of the description of α given above, we see that the elements of $\mathrm{Cl}^0(\Phi)$ are those for which the monomials e_J are products of an even number of factors, and the elements of $\mathrm{Cl}^1(\Phi)$ are those for which the monomials e_J are products of an odd number of factors.

Remark: Observe that $\mathrm{Cl}^0(\Phi)$ is a subalgebra of $\mathrm{Cl}(\Phi)$, whereas $\mathrm{Cl}^1(\Phi)$ is not.

Given two $\mathbb{Z}/2$-graded algebras $A = A^0 \oplus A^1$ and $B = B^0 \oplus B^1$, their *graded tensor product* $A \mathbin{\widehat{\otimes}} B$ is defined by

$$\begin{aligned}(A \mathbin{\widehat{\otimes}} B)^0 &= (A^0 \oplus B^0) \otimes (A^1 \oplus B^1),\\ (A \mathbin{\widehat{\otimes}} B)^1 &= (A^0 \oplus B^1) \otimes (A^1 \oplus B^0),\end{aligned}$$

with multiplication

$$(a' \otimes b)(a \otimes b') = (-1)^{ij}(a'a) \otimes (bb'),$$

for $a \in A^i$ and $b \in B^j$. The reader should check that $A \mathbin{\widehat{\otimes}} B$ is indeed $\mathbb{Z}/2$-graded.

Proposition 21.3 *Let V and W be finite dimensional vector spaces with quadratic forms Φ and Ψ. Then, there is a quadratic form, $\Phi \oplus \Psi$, on $V \oplus W$ defined by*

$$(\Phi + \Psi)(v, w) = \Phi(v) + \Psi(w).$$

If we write $i\colon V \to \mathrm{Cl}(\Phi)$ and $j\colon W \to \mathrm{Cl}(\Psi)$, we can define a linear map,

$$f\colon V \oplus W \to \mathrm{Cl}(\Phi) \mathbin{\widehat{\otimes}} \mathrm{Cl}(\Psi),$$

by

$$f(v, w) = i(v) \otimes 1 + 1 \otimes j(w).$$

Furthermore, the map f induces an isomorphism (also denoted by f)

$$f\colon \mathrm{Cl}(V \oplus W) \to \mathrm{Cl}(\Phi) \mathbin{\widehat{\otimes}} \mathrm{Cl}(\Psi).$$

Proof. See Bröcker and tom Dieck [25], Chapter 1, Section 6, page 57. □

As a corollary, we obtain the following result:

Theorem 21.4 *For every vector space, V, of finite dimension n, the map $i\colon V \to \mathrm{Cl}(\Phi)$ is injective. Given a basis $(e_1, \ldots, e_n)$ of V, the $2^n - 1$ products*

$$i(e_{i_1})i(e_{i_2}) \cdots i(e_{i_k}), \quad 1 \le i_1 < i_2 \cdots < i_k \le n,$$

and 1 form a basis of $\mathrm{Cl}(\Phi)$. Thus, $\mathrm{Cl}(\Phi)$ has dimension 2^n.

Proof. The proof is by induction on $n = \dim(V)$. For $n = 1$, the tensor algebra $T(V)$ is just the polynomial ring $\mathbb{R}[X]$, where $i(e_1) = X$. Thus, $\mathrm{Cl}(\Phi) = \mathbb{R}[X]/(X^2 - \Phi(e_1))$, and the result is obvious. Since

$$i(e_j)i(e_k) + i(e_k)i(e_j) = 2\varphi(e_i, e_j) \cdot 1,$$

it is clear that the products

$$i(e_{i_1})i(e_{i_2}) \cdots i(e_{i_k}), \quad 1 \le i_1 < i_2 \cdots < i_k \le n,$$

and 1 generate $\mathrm{Cl}(\Phi)$. Now, there is always a basis that is orthogonal with respect to φ (for example, see Artin [7], Chapter 7, or Gallier [58], Chapter 6, Problem 6.14), and thus, we have a splitting

$$(V, \Phi) = \bigoplus_{k=1}^{n} (V_k, \Phi_k),$$

where V_k has dimension 1. Choosing a basis so that $e_k \in V_k$, the theorem follows by induction from Proposition 21.3. $\square$

Since i is injective, for simplicity of notation, from now on, we write u for $i(u)$. Theorem 21.4 implies that if $(e_1, \ldots, e_n)$ is an orthogonal basis of V, then $\mathrm{Cl}(\Phi)$ is the algebra presented by the generators $(e_1, \ldots, e_n)$ and the relations

$$\begin{aligned} e_j^2 &= \Phi(e_j) \cdot 1, \quad 1 \leq j \leq n, \quad \text{and} \\ e_j e_k &= -e_k e_j, \quad 1 \leq j, k \leq n, \; j \neq k. \end{aligned}$$

If V has finite dimension n and $(e_1, \ldots, e_n)$ is a basis of V, by Theorem 21.4, the maps t and α are completely determined by their action on the basis elements. Namely, t is defined by

$$\begin{aligned} t(e_i) &= e_i \\ t(e_{i_1} e_{i_2} \cdots e_{i_k}) &= e_{i_k} e_{i_{k-1}} \cdots e_{i_1}, \end{aligned}$$

where $1 \leq i_1 < i_2 \cdots < i_k \leq n$, and, of course, $t(1) = 1$. The map α is defined by

$$\begin{aligned} \alpha(e_i) &= -e_i \\ \alpha(e_{i_1} e_{i_2} \cdots e_{i_k}) &= (-1)^k e_{i_1} e_{i_2} \cdots e_{i_k} \end{aligned}$$

where $1 \leq i_1 < i_2 \cdots < i_k \leq n$, and, of course, $\alpha(1) = 1$. Furthermore, the even-graded elements (the elements of $\mathrm{Cl}^0(\Phi)$) are those generated by 1 and the basis elements consisting of an even number of factors, $e_{i_1} e_{i_2} \cdots e_{i_{2k}}$, and the odd-graded elements (the elements of $\mathrm{Cl}^1(\Phi)$) are those generated by the basis elements consisting of an odd number of factors, $e_{i_1} e_{i_2} \cdots e_{i_{2k+1}}$.

We are now ready to define the Clifford group and investigate some of its properties.

21.3 Clifford Groups

First, we define *conjugation* on a Clifford algebra, $\mathrm{Cl}(\Phi)$, as the map

$$x \mapsto \overline{x} = t(\alpha(x)) \quad \text{for all } x \in \mathrm{Cl}(\Phi).$$

Observe that

$$t \circ \alpha = \alpha \circ t.$$

If V has finite dimension n and $(e_1, \ldots, e_n)$ is a basis of V, in view of previous remarks, conjugation is defined by

$$\begin{aligned}\overline{e_i} &= -e_i\\ \overline{e_{i_1}e_{i_2}\cdots e_{i_k}} &= (-1)^k e_{i_k}e_{i_{k-1}}\cdots e_{i_1}\end{aligned}$$

where $1 \leq i_1 < i_2 \cdots < i_k \leq n$, and, of course, $\overline{1} = 1$. Conjugation is an anti-automorphism.

The multiplicative group of invertible elements of $\mathrm{Cl}(\Phi)$ is denoted by $\mathrm{Cl}(\Phi)^*$.

Definition 21.4 Given a finite dimensional vector space, V, and a quadratic form, Φ, on V, the *Clifford group of* Φ is the group

$$\Gamma(\Phi) = \{x \in \mathrm{Cl}(\Phi)^* \mid \alpha(x)vx^{-1} \in V \quad \text{for all } v \in V\}.$$

The map $N\colon \mathrm{Cl}(Q) \to \mathrm{Cl}(Q)$ given by

$$N(x) = x\overline{x}$$

is called the *norm* of $\mathrm{Cl}(\Phi)$.

We see that the group $\Gamma(\Phi)$ acts on V *via*

$$x \cdot v = \alpha(x)vx^{-1},$$

where $x \in \Gamma(\Phi)$ and $v \in V$. Actually, it is not entirely obvious why the action $\Gamma(\Phi)\times V \longrightarrow V$ is a linear action, and for that matter, why $\Gamma(\Phi)$ is a group.

This is because V is finite-dimensional and α is an automorphism. As a consequence, for any $x \in \Gamma(\Phi)$, the map ρ_x from V to V defined by

$$v \mapsto \alpha(x)vx^{-1}$$

is linear and injective, and thus bijective, since V has finite dimension. It follows that $x^{-1} \in \Gamma(\Phi)$ (the reader should fill in the details).

We also define the group $\Gamma^+(\Phi)$, called the *special Clifford group*, by

$$\Gamma^+(\Phi) = \Gamma(\Phi) \cap \mathrm{Cl}^0(\Phi).$$

Observe that $N(v) = -\Phi(v)\cdot 1$ for all $v \in V$. Also, if $(e_1, \ldots, e_n)$ is a basis of V, we leave it as an exercise to check that

$$N(e_{i_1}e_{i_2}\cdots e_{i_k}) = (-1)^k\Phi(e_{i_1})\Phi(e_{i_2})\cdots\Phi(e_{i_k})\cdot 1.$$

Remark: The map $\rho\colon \Gamma(\Phi) \to \mathbf{GL}(V)$ given by $x \mapsto \rho_x$ is called the *twisted adjoint representation*. It was introduced by Atiyah, Bott and Shapiro [11]. It has the advantage of not introducing a spurious negative sign, i.e., when $v \in V$ and $\Phi(v) \neq 0$, the map ρ_v is the reflection s_v about the hyperplane orthogonal to v (see Proposition 21.6). Furthermore, when Φ is nondegenerate, the kernel $\mathrm{Ker}\,(\rho)$ of the representation ρ is given by $\mathrm{Ker}\,(\rho) = \mathbb{R}^* \cdot 1$, where $\mathbb{R}^* = \mathbb{R} - \{0\}$. The earlier *adjoint representation* (used by Chevalley [35] and others) is given by

$$v \mapsto xvx^{-1}.$$

Unfortunately, in this case, ρ_x represents $-s_v$, where s_v is the reflection about the hyperplane orthogonal to v. Furthermore, the kernel of the representation ρ is generally bigger than $\mathbb{R}^* \cdot 1$. This is the reason why the twisted adjoint representation is preferred (and must be used for a proper treatment of the **Pin** group).

Proposition 21.5 *The maps α and t induce an automorphism and an anti-automorphism of the Clifford group, $\Gamma(\Phi)$.*

Proof. It is not very instructive, see Bröcker and tom Dieck [25], Chapter 1, Section 6, page 58.

The following proposition shows why Clifford groups generalize the quaternions.

Proposition 21.6 *Let V be a finite dimensional vector space and Φ a quadratic form on V. For every element, x, of the Clifford group, $\Gamma(\Phi)$, if $\Phi(x) \neq 0$, then the map $\rho_x \colon V \to V$ given by*

$$v \mapsto \alpha(x)vx^{-1} \quad \textit{for all } v \in V$$

is the reflection about the hyperplane H orthogonal to the vector x.

Proof. Recall that the reflection s about the hyperplane H orthogonal to the vector x is given by

$$s(u) = u - 2\,\frac{\varphi(u,x)}{\Phi(x)} \cdot x.$$

However, we have

$$x^2 = \Phi(x) \cdot 1 \quad \text{and} \quad ux + xu = 2\varphi(u,x) \cdot 1.$$

Thus, we have

$$\begin{aligned}
s(u) &= u - 2\,\frac{\varphi(u,x)}{\Phi(x)} \cdot x \\
&= u - 2\varphi(u,x) \cdot \left(\frac{1}{\Phi(x)} \cdot x\right) \\
&= u - 2\varphi(u,x) \cdot x^{-1} \\
&= u - 2\varphi(u,x) \cdot (1x^{-1}) \\
&= u - (2\varphi(u,x) \cdot 1)x^{-1} \\
&= u - (ux + xu)x^{-1} \\
&= -xux^{-1} \\
&= \alpha(x)ux^{-1},
\end{aligned}$$

since $\alpha(x) = -x$, for $x \in V$. $\square$

In general, we have a map

$$\rho\colon \Gamma(\Phi) \to \mathbf{GL}(V)$$

defined by

$$\rho(x)(v) = \alpha(x)vx^{-1},$$

for all $x \in \Gamma(\Phi)$ and all $v \in V$. We would like to show that ρ is a surjective homomorphism from $\Gamma(\Phi)$ onto $\mathbf{O}(\varphi)$ and a surjective homomorphism from $\Gamma^+(\Phi)$ onto $\mathbf{SO}(\varphi)$. For this, we will need to assume that φ is nondegenerate, which means that for every $v \in V$, if $\varphi(v, w) = 0$ for all $w \in V$, then $v = 0$. For simplicity of exposition, we first assume that Φ is the quadratic form on $\mathbb{R}^n$ defined by

$$\Phi(x_1, \ldots, x_n) = -(x_1^2 + \cdots + x_n^2).$$

Let Cl_n denote the Clifford algebra $\mathrm{Cl}(\Phi)$ and Γ_n denote the Clifford group $\Gamma(\Phi)$. The following lemma plays a crucial role:

Lemma 21.7 *The kernel of the map $\rho\colon \Gamma_n \to \mathbf{GL}(n)$ is $\mathbb{R}^* \cdot 1$, the multiplicative group of nonzero scalar multiples of $1 \in \mathrm{Cl}_n$.*

Proof. If $\rho(x) = \mathrm{id}$, then

$$\alpha(x)v = vx \quad \text{for all } v \in \mathbb{R}^n. \tag{1}$$

Since $\mathrm{Cl}_n = \mathrm{Cl}_n^0 \oplus \mathrm{Cl}_n^1$, we can write $x = x^0 + x^1$, with $x^i \in \mathrm{Cl}_n^i$ for $i = 1, 2$. Then, equation (1) becomes

$$x^0v = vx^0 \quad \text{and} \quad -x^1v = vx^1 \quad \text{for all } v \in \mathbb{R}^n. \tag{2}$$

Using Theorem 21.4, we can express x^0 as a linear combination of monomials in the canonical basis $(e_1, \ldots, e_n)$, so that

$$x^0 = a^0 + e_1b^1, \quad \text{with } a^0 \in \mathrm{Cl}_n^0,\ b^1 \in \mathrm{Cl}_n^1,$$

where neither a^0 nor b^1 contains a summand with a factor e_1. Applying the first relation in (2) to $v = e_1$, we get

$$e_1a^0 + e_1^2b^1 = a^0e_1 + e_1b^1e_1. \tag{3}$$

Now, the basis $(e_1, \ldots, e_n)$ is orthogonal w.r.t. Φ, which implies that

$$e_je_k = -e_ke_j \quad \text{for all } j \neq k.$$

Since each monomial in a^0 is of even degre and contains no factor e_1, we get

$$a^0e_1 = e_1a^0.$$

Similarly, since b^1 is of odd degree and contains no factor e_1, we get

$$e_1 b^1 e_1 = -e_1^2 b^1.$$

But then, from (3), we get

$$e_1 a^0 + e_1^2 b^1 = a^0 e_1 + e_1 b^1 e_1 = e_1 a^0 - e_1^2 b^1,$$

and so, $e_1^2 b^1 = 0$. However, $e_1^2 = -1$, and so, $b_1 = 0$. Therefore, x_0 contains no monomial with a factor e_1. We can apply the same argument to the other basis elements $e_2, \ldots, e_n$, and thus, we just proved that $x^0 \in \mathbb{R} \cdot 1$.

A similar argument applying to the second equation in (2), with $x^1 = a^1 + e_1 b^0$ and $v = e_1$ shows that $b^0 = 0$. We also conclude that $x^1 \in \mathbb{R} \cdot 1$. However, $\mathbb{R} \cdot 1 \subseteq \mathrm{Cl}_n^0$, and so, $x^1 = 0$. Finally, $x = x^0 \in (\mathbb{R} \cdot 1) \cap \Gamma_n = \mathbb{R}^* \cdot 1$. $\square$

Remark: If Φ is any nondegenerate quadratic form, we know (for instance, see Artin [7], Chapter 7, or Gallier [58], Chapter 6, Problem 6.14) that there is an orthogonal basis $(e_1, \ldots, e_n)$ with respect to φ (i.e. $\varphi(e_j, e_k) = 0$ for all $j \neq k$). Thus, the commutation relations

$$\begin{aligned} e_j^2 &= \Phi(e_j) \cdot 1, \quad \text{with } \Phi(e_j) \neq 0, \quad 1 \leq j \leq n, \quad \text{and} \\ e_j e_k &= -e_k e_j, \quad 1 \leq j, k \leq n,\ j \neq k \end{aligned}$$

hold, and since the proof only rests on these facts, Lemma 21.7 holds for any nondegenerate quadratic form.

However, Lemma 21.7 may fail for degenerate quadratic forms. For example, if $\Phi \equiv 0$, then $\mathrm{Cl}(V, 0) = \bigwedge^{\bullet} V$. Consider the element $x = 1 + e_1 e_2$. Clearly, $x^{-1} = 1 - e_1 e_2$. But now, for any $v \in V$, we have

$$\alpha(1 + e_1 e_2) v (1 + e_1 e_2)^{-1} = (1 + e_1 e_2) v (1 - e_1 e_2) = v.$$

Yet, $1 + e_1 e_2$ is not a scalar multiple of 1.

The following proposition shows that the notion of norm is well-behaved.

Proposition 21.8 *If $x \in \Gamma_n$, then $N(x) \in \mathbb{R}^* \cdot 1$.*

Proof. The trick is to show that $N(x)$ is in the kernel of ρ. To say that $x \in \Gamma_n$ means that

$$\alpha(x) v x^{-1} \in \mathbb{R}^n \quad \text{for all } v \in \mathbb{R}^n.$$

Applying t, we get

$$t(x)^{-1} v t(\alpha(x)) = \alpha(x) v x^{-1},$$

since t is the identity on $\mathbb{R}^n$. Thus, we have

$$v = t(x)\alpha(x)v(t(\alpha(x))x)^{-1} = \alpha(\overline{x}x)v(\overline{x}x)^{-1},$$

so $\overline{x}x \in \mathrm{Ker}\,(\rho)$. By Proposition 21.5, we have $\overline{x} \in \Gamma_n$, and so, $x\overline{x} = \overline{\overline{x}}\,\overline{x} \in \mathrm{Ker}\,(\rho)$. $\square$

Remark: Again, the proof also holds for the Clifford group $\Gamma(\Phi)$ associated with any nondegenerate quadratic form Φ. When $\Phi(v) = -\|v\|^2$, where $\|v\|$ is the standard Euclidean norm of v, we have $N(v) = \|v\|^2 \cdot 1$ for all $v \in V$. However, for other quadratic forms, it is possible that $N(x) = \lambda \cdot 1$ where $\lambda < 0$, and this is a difficulty that needs to be overcome.

Proposition 21.9 *The restriction of the norm, N, to Γ_n is a homomorphism, $N\colon \Gamma_n \to \mathbb{R}^* \cdot 1$, and $N(\alpha(x)) = N(x)$ for all $x \in \Gamma_n$.*

Proof. We have

$$N(xy) = xy\overline{y}\,\overline{x} = xN(y)\overline{x} = x\overline{x}N(y) = N(x)N(y),$$

where the third equality holds because $N(x) \in \mathbb{R}^* \cdot 1$. We also have

$$N(\alpha(x)) = \alpha(x)\alpha(\overline{x}) = \alpha(x\overline{x}) = \alpha(N(x)) = N(x).$$

$\square$

Remark: The proof also holds for the Clifford group $\Gamma(\Phi)$ associated with any nondegenerate quadratic form Φ.

Proposition 21.10 *We have $\mathbb{R}^n - \{0\} \subseteq \Gamma_n$ and $\rho(\Gamma_n) \subseteq \mathbf{O}(n)$.*

Proof. Let $x \in \Gamma_n$ and $v \in \mathbb{R}^n$, with $v \neq 0$. We have

$$N(\rho(x)(v)) = N(\alpha(x)vx^{-1}) = N(\alpha(x))N(v)N(x^{-1}) = N(x)N(v)N(x)^{-1} = N(v),$$

since $N\colon \Gamma_n \to \mathbb{R}^* \cdot 1$. However, for $v \in \mathbb{R}^n$, we know that

$$N(v) = -\Phi(v) \cdot 1.$$

Thus, $\rho(x)$ is norm-preserving, and so, $\rho(x) \in \mathbf{O}(n)$. $\square$

Remark: The proof that $\rho(\Gamma(\Phi)) \subseteq \mathbf{O}(\Phi)$ also holds for the Clifford group $\Gamma(\Phi)$ associated with any nondegenerate quadratic form Φ. The first statement needs to be replaced by the fact that every non-isotropic vector in $\mathbb{R}^n$ (a vector is non-isotropic if $\Phi(x) \neq 0$) belongs to $\Gamma(\Phi)$. Indeed, $x^2 = \Phi(x) \cdot 1$, which implies that x is invertible.

We are finally ready for the introduction of the groups $\mathbf{Pin}(n)$ and $\mathbf{Spin}(n)$.

21.4 The Groups $\mathbf{Pin}(n)$ and $\mathbf{Spin}(n)$

Definition 21.5 We define the *pinor group,* $\mathbf{Pin}(n)$*,* as the kernel $\mathrm{Ker}\,(N)$ of the homomorphism $N\colon \Gamma_n \to \mathbb{R}^* \cdot 1$, and the *spinor group,* $\mathbf{Spin}(n)$*,* as $\mathbf{Pin}(n) \cap \Gamma_n^+$.

Observe that if $N(x) = 1$, then x is invertible and $x^{-1} = \overline{x}$, since $x\overline{x} = N(x) = 1$. Thus, we can write

$$\mathbf{Pin}(n) = \{x \in \mathrm{Cl}_n \mid xvx^{-1} \in \mathbb{R}^n \quad \text{for all } v \in \mathbb{R}^n, \quad N(x) = 1\},$$

and

$$\mathbf{Spin}(n) = \{x \in \mathrm{Cl}_n^0 \mid xvx^{-1} \in \mathbb{R}^n \quad \text{for all } v \in \mathbb{R}^n, \quad N(x) = 1\}.$$

Remark: According to Atiyah, Bott and Shapiro, the use of the name $\mathbf{Pin}(k)$ is a joke due to Jean-Pierre Serre (Atiyah, Bott and Shapiro [11], page 1).

Theorem 21.11 *The restriction of* ρ *to the pinor group,* $\mathbf{Pin}(n)$*, is a surjective homomorphism,* $\rho\colon \mathbf{Pin}(n) \to \mathbf{O}(n)$*, whose kernel is* $\{-1, 1\}$*, and the restriction of* ρ *to the spinor group,* $\mathbf{Spin}(n)$*, is a surjective homomorphism,* $\rho\colon \mathbf{Spin}(n) \to \mathbf{SO}(n)$*, whose kernel is* $\{-1, 1\}$.

Proof. By Proposition 21.10, we have a map $\rho\colon \mathbf{Pin}(n) \to \mathbf{O}(n)$. The reader can easily check that ρ is a homomorphism. By the Cartan-Dieudonné theorem (see Bourbaki [20], or Gallier [58], Chapter 7, Theorem 7.2.1), every isometry $f \in \mathbf{SO}(n)$ is the composition $f = s_1 \circ \cdots \circ s_k$ of hyperplane reflections s_j. If we assume that s_j is a reflection about the hyperplane H_j orthogonal to the nonzero vector w_j, by Proposition 21.6, $\rho(w_j) = s_j$. Since $N(w_j) = \|w_j\|^2 \cdot 1$, we can replace w_j by $w_j / \|w_j\|$, so that $N(w_1 \cdots w_k) = 1$, and then

$$f = \rho(w_1 \cdots w_k),$$

and ρ is surjective. Note that

$$\mathrm{Ker}\,(\rho \mid \mathbf{Pin}(n)) = \mathrm{Ker}\,(\rho) \cap \ker(N) = \{t \in \mathbb{R}^* \cdot 1 \mid N(t) = 1\} = \{-1, 1\}.$$

As to $\mathbf{Spin}(n)$, we just need to show that the restriction of ρ to $\mathbf{Spin}(n)$ maps Γ_n into $\mathbf{SO}(n)$. If this was not the case, there would be some improper isometry $f \in \mathbf{O}(n)$ so that $\rho(x) = f$, where $x \in \Gamma_n \cap \mathrm{Cl}_n^0$. However, we can express f as the composition of an odd number of reflections, say

$$f = \rho(w_1 \cdots w_{2k+1}).$$

Since

$$\rho(w_1 \cdots w_{2k+1}) = \rho(x),$$

we have $x^{-1} w_1 \cdots w_{2k+1} \in \mathrm{Ker}\,(\rho)$. By Lemma 21.7, we must have

$$x^{-1} w_1 \cdots w_{2k+1} = \lambda \cdot 1$$

for some $\lambda \in \mathbb{R}^*$, and thus,

$$w_1 \cdots w_{2k+1} = \lambda \cdot x,$$

where x has even degree and $w_1 \cdots w_{2k+1}$ has odd degree, which is impossible. $\square$

Let us denote the set of elements $v \in \mathbb{R}^n$ with $N(v) = 1$ (with norm 1) by S^{n-1}. We have the following corollary of Theorem 21.11:

Corollary 21.12 *The group* $\mathbf{Pin}(n)$ *is generated by* S^{n-1} *and every element of* $\mathbf{Spin}(n)$ *can be written as the product of an even number of elements of* S^{n-1}.

Example 21.2 The reader should verify that

$$\mathbf{Pin}(1) \approx \mathbb{Z}/4\mathbb{Z}, \quad \mathbf{Spin}(1) = \{-1, 1\} \approx \mathbb{Z}/2\mathbb{Z},$$

and also that

$$\mathbf{Pin}(2) \approx \{ae_1 + be_2 \mid a^2 + b^2 = 1\} \cup \{c1 + de_1e_2 \mid c^2 + d^2 = 1\}, \quad \mathbf{Spin}(2) = \mathbf{U}(1).$$

We may also write $\mathbf{Pin}(2) = \mathbf{U}(1) + \mathbf{U}(1)$, where $\mathbf{U}(1)$ is the group of complex numbers of modulus 1 (the unit circle in $\mathbb{R}^2$). It can also be shown that $\mathbf{Spin}(3) \approx \mathbf{SU}(2)$ and $\mathbf{Spin}(4) \approx \mathbf{SU}(2) \times \mathbf{SU}(2)$. The group $\mathbf{Spin}(5)$ is isomorphic to the symplectic group $\mathbf{Sp}(2)$, and $\mathbf{Spin}(6)$ is isomorphic to $\mathbf{SU}(4)$ (see Curtis [38] or Porteous [124]).

Let us take a closer look at $\mathbf{Spin}(2)$. The Clifford algebra Cl_2 is generated by the four elements

$$1,\ e_1,\ e_2,\ , e_1e_2,$$

and they satisfy the relations

$$e_1^2 = -1, \quad e_2^2 = -1, \quad e_1e_2 = -e_2e_1.$$

The group $\mathbf{Spin}(2)$ consists of all products

$$\prod_{i=1}^{2k} (a_ie_1 + b_ie_2)$$

consisting of an even number of factors and such that $a_i^2 + b_i^2 = 1$. In view of the above relations, every such element can be written as

$$x = a1 + be_1e_2,$$

where x satisfies the conditions that $xvx^{-1} \in \mathbb{R}^2$ for all $v \in \mathbb{R}^2$, and $N(x) = 1$. Since

$$\overline{X} = a1 - be_1e_2,$$

we get

$$N(x) = a^2 + b^2,$$

and the condition $N(x) = 1$ is simply $a^2 + b^2 = 1$. We claim that $xvx^{-1} \in \mathbb{R}^2$ if $x \in \mathrm{Cl}_2^0$. Indeed, since $x \in \mathrm{Cl}_2^0$ and $v \in \mathrm{Cl}_2^1$, we have $xvx^{-1} \in \mathrm{Cl}_2^1$, which implies that $xvx^{-1} \in \mathbb{R}^2$, since the only elements of Cl_2^1 are those in $\mathbb{R}^2$. Then, $\mathbf{Spin}(2)$ consists of those elements $x = a1 + be_1e_2$ so that $a^2 + b^2 = 1$. If we let $\mathbf{i} = e_1e_2$, we observe that

$$\begin{aligned} \mathbf{i}^2 &= -1, \\ e_1\mathbf{i} &= -\mathbf{i}e_1 = -e_2, \\ e_2\mathbf{i} &= -\mathbf{i}e_2 = e_1. \end{aligned}$$

Thus, $\mathbf{Spin}(2)$ is isomorphic to $\mathbf{U}(1)$. Also note that

$$e_1(a1 + b\mathbf{i}) = (a1 - b\mathbf{i})e_1.$$

Let us find out explicitly what is the action of $\mathbf{Spin}(2)$ on $\mathbb{R}^2$. Given $X = a1 + b\mathbf{i}$, with $a^2 + b^2 = 1$, for any $v = v_1e_1 + v_2e_2$, we have

$$\begin{aligned} \alpha(X)vX^{-1} &= X(v_1e_1 + v_2e_2)X^{-1} \\ &= X(v_1e_1 + v_2e_2)(-e_1e_1)\overline{X} \\ &= X(v_1e_1 + v_2e_2)(-e_1)(e_1\overline{X}) \\ &= X(v_11 + v_2\mathbf{i})Xe_1 \\ &= X^2(v_11 + v_2\mathbf{i})e_1 \\ &= (((a^2 - b^2)v_1 - 2abv_2)1 + (a^2 - b^2)v_2 + 2abv_1)\mathbf{i})e_1 \\ &= ((a^2 - b^2)v_1 - 2abv_2)e_1 + (a^2 - b^2)v_2 + 2abv_1)e_2. \end{aligned}$$

Since $a^2 + b^2 = 1$, we can write $X = a1 + b\mathbf{i} = (\cos\theta)1 + (\sin\theta)\mathbf{i}$, and the above derivation shows that

$$\alpha(X)vX^{-1} = (\cos 2\theta v_1 - \sin 2\theta v_2)e_1 + (\cos 2\theta v_2 + \sin 2\theta v_1)e_2.$$

This means that the rotation ρ_X induced by $X \in \mathbf{Spin}(2)$ is the rotation of angle 2θ around the origin. Observe that the maps

$$v \mapsto v(-e_1), \quad X \mapsto Xe_1$$

establish bijections between $\mathbb{R}^2$ and $\mathbf{Spin}(2) \simeq \mathbf{U}(1)$. Also, note that the action of $X = \cos\theta + i\sin\theta$ viewed as a complex number yields the rotation of angle θ, whereas the action of $X = (\cos\theta)1 + (\sin\theta)\mathbf{i}$ viewed as a member of $\mathbf{Spin}(2)$ yields the rotation of angle 2θ. There is nothing wrong. In general, $\mathbf{Spin}(n)$ is a two–to–one cover of $\mathbf{SO}(n)$.

Next, let us take a closer look at $\mathbf{Spin}(3)$. The Clifford algebra Cl_3 is generated by the eight elements

$$1,\ e_1,\ e_2,\ ,e_3,\ ,e_1e_2,\ e_2e_3,\ e_3e_1,\ e_1e_2e_3,$$

and they satisfy the relations

$$e_i^2 = -1, \quad e_j e_j = -e_j e_i, \quad 1 \leq i, j \leq 3, \; i \neq j.$$

The group **Spin**(3) consists of all products

$$\prod_{i=1}^{2k} (a_i e_1 + b_i e_2 + c_i e_3)$$

consisting of an even number of factors and such that $a_i^2 + b_i^2 + c_i^2 = 1$. In view of the above relations, every such element can be written as

$$x = a1 + be_2e_3 + ce_3e_1 + de_1e_2,$$

where x satisfies the conditions that $xvx^{-1} \in \mathbb{R}^3$ for all $v \in \mathbb{R}^3$, and $N(x) = 1$. Since

$$\overline{X} = a1 - be_2e_3 - ce_3e_1 - de_1e_2,$$

we get

$$N(x) = a^2 + b^2 + c^2 + d^2,$$

and the condition $N(x) = 1$ is simply $a^2 + b^2 + c^2 + d^2 = 1$.

It turns out that the conditions $x \in \mathrm{Cl}_3^0$ and $N(x) = 1$ imply that $xvx^{-1} \in \mathbb{R}^3$ for all $v \in \mathbb{R}^3$. To prove this, first observe that $N(x) = 1$ implies that $x^{-1} = \pm\overline{x}$, and that $\overline{v} = -v$ for any $v \in \mathbb{R}^3$, and so,

$$\overline{xvx^{-1}} = -xvx^{-1}.$$

Also, since $x \in \mathrm{Cl}_3^0$ and $v \in \mathrm{Cl}_3^1$, we have $xvx^{-1} \in \mathrm{Cl}_3^1$. Thus, we can write

$$xvx^{-1} = u + \lambda e_1e_2e_3, \quad \text{for some } u \in \mathbb{R}^3 \text{ and some } \lambda \in \mathbb{R}.$$

But

$$\overline{e_1e_2e_3} = -e_3e_2e_1 = e_1e_2e_3,$$

and so,

$$\overline{xvx^{-1}} = -u + \lambda e_1e_2e_3 = -xvx^{-1} = -u - \lambda e_1e_2e_3,$$

which implies that $\lambda = 0$. Thus, $xvx^{-1} \in \mathbb{R}^3$, as claimed. Then, **Spin**(3) consists of those elements $x = a1 + be_2e_3 + ce_3e_1 + de_1e_2$ so that $a^2 + b^2 + c^2 + d^2 = 1$. Under the bijection

$$\mathbf{i} \mapsto e_2e_3, \; \mathbf{j} \mapsto e_3e_1, \; \mathbf{k} \mapsto e_1e_2,$$

we can check that we have an isomorphism between the group **SU**(2) of unit quaternions and **Spin**(3). If $X = a1 + be_2e_3 + ce_3e_1 + de_1e_2 \in \mathbf{Spin}(3)$, observe that

$$X^{-1} = \overline{X} = a1 - be_2e_3 - ce_3e_1 - de_1e_2.$$

Now, using the identification

$$\mathbf{i} \mapsto e_2e_3,\ \mathbf{j} \mapsto e_3e_1,\ \mathbf{k} \mapsto e_1e_2,$$

we can easily check that

$$\begin{aligned}
(e_1e_2e_3)^2 &= 1,\\
(e_1e_2e_3)\mathbf{i} &= \mathbf{i}(e_1e_2e_3) \;=\; -e_1,\\
(e_1e_2e_3)\mathbf{j} &= \mathbf{j}(e_1e_2e_3) \;=\; -e_2,\\
(e_1e_2e_3)\mathbf{k} &= \mathbf{k}(e_1e_2e_3) \;=\; -e_3,\\
(e_1e_2e_3)e_1 &= -\mathbf{i},\\
(e_1e_2e_3)e_2 &= -\mathbf{j},\\
(e_1e_2e_3)e_3 &= -\mathbf{k}.
\end{aligned}$$

Then, if $X = a1 + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbf{Spin}(3)$, for every $v = v_1e_1 + v_2e_2 + v_3e_3$, we have

$$\begin{aligned}
\alpha(X)vX^{-1} &= X(v_1e_1 + v_2e_2 + v_3e_3)X^{-1}\\
&= X(e_1e_2e_3)^2(v_1e_1 + v_2e_2 + v_3e_3)X^{-1}\\
&= (e_1e_2e_3)X(e_1e_2e_3)(v_1e_1 + v_2e_2 + v_3e_3)X^{-1}\\
&= -(e_1e_2e_3)X(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})X^{-1}.
\end{aligned}$$

This shows that the rotation $\rho_X \in \mathbf{SO}(3)$ induced by $X \in \mathbf{Spin}(3)$ can be viewed as the rotation induced by the quaternion $a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ on the pure quaternions, using the maps

$$v \mapsto -(e_1e_2e_3)v, \quad X \mapsto -(e_1e_2e_3)X$$

to go from a vector $v = v_1e_1 + v_2e_2 + v_3e_3$ to the pure quaternion $v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and back.

We close this section by taking a closer look at $\mathbf{Spin}(4)$. The group $\mathbf{Spin}(4)$ consists of all products

$$\prod_{i=1}^{2k}(a_ie_1 + b_ie_2 + c_ie_3 + d_ie_4)$$

consisting of an even number of factors and such that $a_i^2 + b_i^2 + c_i^2 + d_i^2 = 1$. Using the relations

$$e_i^2 = -1, \quad e_je_j = -e_je_i, \quad 1 \leq i, j \leq 4,\ i \neq j,$$

every element of $\mathbf{Spin}(4)$ can be written as

$$x = a_11 + a_2e_1e_2 + a_3e_2e_3 + a_4e_3e_1 + a_5e_4e_3 + a_6e_4e_1 + a_7e_4e_2 + a_8e_1e_2e_3e_4,$$

where x satisfies the conditions that $xvx^{-1} \in \mathbb{R}^4$ for all $v \in \mathbb{R}^4$, and $N(x) = 1$. Let

$$\mathbf{i} = e_1e_2,\ \mathbf{j} = e_2e_3,\ \mathbf{k} = 3_3e_1,\ \mathbf{i}' = e_4e_3,\ \mathbf{j}' = e_4e_1,\ \mathbf{k}' = e_4e_2,$$

and $\mathbb{I} = e_1e_2e_3e_4$. The reader will easily verify that

$$\begin{aligned}
\mathbf{ij} &= \mathbf{k} \\
\mathbf{jk} &= \mathbf{i} \\
\mathbf{ki} &= \mathbf{j} \\
\mathbf{i}^2 &= -1, \quad \mathbf{j}^2 = -1, \quad \mathbf{k}^2 = -1 \\
\mathbf{i}\mathbb{I} &= \mathbb{I}\mathbf{i} = \mathbf{i}' \\
\mathbf{j}\mathbb{I} &= \mathbb{I}\mathbf{j} = \mathbf{j}' \\
\mathbf{k}\mathbb{I} &= \mathbb{I}\mathbf{k} = \mathbf{k}' \\
\mathbb{I}^2 &= 1, \quad \overline{\mathbb{I}} = \mathbb{I}.
\end{aligned}$$

Then, every $x \in \mathbf{Spin}(4)$ can be written as

$$x = u + \mathbb{I}v, \quad \text{with} \quad u = a1 + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \quad \text{and} \quad v = a'1 + b'\mathbf{i} + c'\mathbf{j} + d'\mathbf{k},$$

with the extra conditions stated above. Using the above identities, we have

$$(u + \mathbb{I}v)(u' + \mathbb{I}v') = uu' + vv' + \mathbb{I}(uv' + vu').$$

As a consequence,

$$N(u + \mathbb{I}v) = (u + \mathbb{I}v)(\overline{u} + \mathbb{I}\overline{v}) = u\overline{u} + v\overline{v} + \mathbb{I}(u\overline{v} + v\overline{u}),$$

and thus, $N(u + \mathbb{I}v) = 1$ is equivalent to

$$u\overline{u} + v\overline{v} = 1 \quad \text{and} \quad u\overline{v} + v\overline{u} = 0.$$

As in the case $n = 3$, it turns out that the conditions $x \in \mathrm{Cl}_4^0$ and $N(x) = 1$ imply that $xvx^{-1} \in \mathbb{R}^4$ for all $v \in \mathbb{R}^4$. The only change to the proof is that $xvx^{-1} \in \mathrm{Cl}_4^1$ can be written as

$$xvx^{-1} = u + \sum_{i,j,k} \lambda_{i,j,k} e_i e_j e_k, \quad \text{for some } u \in \mathbb{R}^4, \quad \text{with } \{i, j, k\} \subseteq \{1, 2, 3, 4\}.$$

As in the previous proof, we get $\lambda_{i,j,k} = 0$. Then, $\mathbf{Spin}(4)$ consists of those elements $u + \mathbb{I}v$ so that

$$u\overline{u} + v\overline{v} = 1 \quad \text{and} \quad u\overline{v} + v\overline{u} = 0,$$

with u and v of the form $a1 + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. Finally, we see that $\mathbf{Spin}(4)$ is isomorphic to $\mathbf{Spin}(2) \times \mathbf{Spin}(2)$ under the isomorphism

$$u + v\mathbb{I} \mapsto (u + v, u - v).$$

Indeed, we have

$$N(u + v) = (u + v)(\overline{u} + \overline{v}) = 1,$$

and

$$N(u-v) = (u-v)(\overline{u}-\overline{v}) = 1,$$

since

$$u\overline{u} + v\overline{v} = 1 \quad \text{and} \quad u\overline{v} + v\overline{u} = 0,$$

and

$$(u+v, u-v)(u'+v', u'-v') = (uu'+vv'+uv'+vu', uu'+vv'-(uv'+vu')).$$

Remark: It can be shown that the assertion if $x \in \mathrm{Cl}_n^0$ and $N(x) = 1$, then $xvx^{-1} \in \mathbb{R}^n$ for all $v \in \mathbb{R}^n$, is true up to $n = 5$ (see Porteous [124], Chapter 13, Proposition 13.58). However, this is already false for $n = 6$. For example, if $X = 1/\sqrt{2}(1 + e_1e_2e_3e_4e_5e_6)$, it is easy to see that $N(X) = 1$, and yet, $Xe_1X^{-1} \notin \mathbb{R}^6$.

21.5 The Groups Pin(p,q) and Spin(p,q)

For every nondegenerate quadratic form Φ over $\mathbb{R}$, there is an orthogonal basis with respect to which Φ is given by

$$\Phi(x_1, \ldots, x_{p+q}) = x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2),$$

where p and q only depend on Φ. The quadratic form corresponding to (p,q) is denoted $\Phi_{p,q}$ and we call (p,q) the *signature of* $\Phi_{p,q}$. Let $n = p+q$. We define the group $\mathbf{O}(p,q)$ as the group of isometries w.r.t. $\Phi_{p,q}$, i.e., the group of linear maps f so that

$$\Phi_{p,q}(f(v)) = \Phi_{p,q}(v) \quad \text{for all } v \in \mathbb{R}^n$$

and the group $\mathbf{SO}(p,q)$ as the subgroup of $\mathbf{O}(p,q)$ consisting of the isometries, $f \in \mathbf{O}(p,q)$, with $\det(f) = 1$. We denote the Clifford algebra $\mathrm{Cl}(\Phi_{p,q})$ where $\Phi_{p,q}$ has signature (p,q) by $\mathrm{Cl}_{p,q}$, the corresponding Clifford group by $\Gamma_{p,q}$, and the special Clifford group $\Gamma_{p,q} \cap \mathrm{Cl}_{p,q}^0$ by $\Gamma_{p,q}^+$. Note that with this new notation, $\mathrm{Cl}_n = \mathrm{Cl}_{0,n}$.

As we mentioned earlier, since Lawson and Michelsohn [96] adopt the opposite of our sign convention in defining Clifford algebras, their $\mathrm{Cl}(p,q)$ is our $\mathrm{Cl}(q,p)$.

As we mentioned in Section 21.3, we have the problem that $N(v) = -\Phi(v) \cdot 1$ but $-\Phi(v)$ is not necessarily positive (where $v \in \mathbb{R}^n$). The fix is simple: Allow elements $x \in \Gamma_{p,q}$ with $N(x) = \pm 1$.

Definition 21.6 We define the *pinor group,* $\mathbf{Pin}(p,q)$*,* as the group

$$\mathbf{Pin}(p,q) = \{x \in \Gamma_{p,q} \mid N(x) = \pm 1\},$$

and the *spinor group,* $\mathbf{Spin}(p,q)$*,* as $\mathbf{Pin}(p,q) \cap \Gamma_{p,q}^+$.

Remarks:

(1) It is easily checked that the group $\mathbf{Spin}(p, q)$ is also given by

$$\mathbf{Spin}(p, q) = \{x \in \mathrm{Cl}^0_{p,q} \mid xv\overline{x} \in \mathbb{R}^n \quad \text{for all } v \in \mathbb{R}^n, \quad N(x) = 1\}.$$

This is because $\mathbf{Spin}(p, q)$ consists of elements of even degree.

(2) One can check that if $N(x) \neq 0$, then

$$\alpha(x)vx^{-1} = xvt(x)/N(x).$$

Thus, we have

$$\mathbf{Pin}(p, q) = \{x \in \mathrm{Cl}_{p,q} \mid xvt(x)N(x) \in \mathbb{R}^n \quad \text{for all } v \in \mathbb{R}^n, \quad N(x) = \pm 1\}.$$

When $\Phi(x) = -\left\|x\right\|^2$, we have $N(x) = \left\|x\right\|^2$, and

$$\mathbf{Pin}(n) = \{x \in \mathrm{Cl}_n \mid xvt(x) \in \mathbb{R}^n \quad \text{for all } v \in \mathbb{R}^n, \quad N(x) = 1\}.$$

Theorem 21.11 generalizes as follows:

Theorem 21.13 *The restriction of ρ to the pinor group, $\mathbf{Pin}(p, q)$, is a surjective homomorphism, $\rho\colon \mathbf{Pin}(p, q) \to \mathbf{O}(p, q)$, whose kernel is $\{-1, 1\}$, and the restriction of ρ to the spinor group, $\mathbf{Spin}(p, q)$, is a surjective homomorphism, $\rho\colon \mathbf{Spin}(p, q) \to \mathbf{SO}(p, q)$, whose kernel is $\{-1, 1\}$.*

Proof. The Cartan-Dieudonné also holds for any nondegenerate quadratic form Φ, in the sense that every isometry in $\mathbf{O}(\Phi)$ is the composition of reflections defined by hyperplanes orthogonal to non-isotropic vectors (see Dieudonné [42], Chevalley [35], Bourbaki [20], or Gallier [58], Chapter 7, Problem 7.14). Thus, Theorem 21.11 also holds for any nondegenerate quadratic form Φ. The only change to the proof is the following: Since $N(w_j) = -\Phi(w_j) \cdot 1$, we can replace w_j by $w_j/\sqrt{|\Phi(w_j)|}$, so that $N(w_1 \cdots w_k) = \pm 1$, and then

$$f = \rho(w_1 \cdots w_k),$$

and ρ is surjective. $\square$

If we consider $\mathbb{R}^n$ equipped with the quadratic form $\Phi_{p,q}$ (with $n = p + q$), we denote the set of elements $v \in \mathbb{R}^n$ with $N(v) = 1$ by $S^{n-1}_{p,q}$. We have the following corollary of Theorem 21.13 (generalizing Corollary 21.14):

Corollary 21.14 *The group $\mathbf{Pin}(p, q)$ is generated by $S^{n-1}_{p,q}$ and every element of $\mathbf{Spin}(p, q)$ can be written as the product of an even number of elements of $S^{n-1}_{p,q}$.*

Example 21.3 The reader should check that

$$\mathrm{Cl}_{0,1} \approx \mathbb{C}, \quad \mathrm{Cl}_{1,0} \approx \mathbb{R} \oplus \mathbb{R}.$$

We also have

$$\mathbf{Pin}(0,1) \approx \mathbb{Z}/4\mathbb{Z}, \quad \mathbf{Pin}(1,0) \approx \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

from which we get $\mathbf{Spin}(0,1) = \mathbf{Spin}(1,0) \approx \mathbb{Z}/2\mathbb{Z}$. Also, show that

$$\mathrm{Cl}_{0,2} \approx \mathbb{H}, \quad \mathrm{Cl}_{1,1} \approx M_2(\mathbb{R}), \quad \mathrm{Cl}_{2,0} \approx M_2(\mathbb{R}),$$

where $M_n(\mathbb{R})$ denotes the algebra of $n \times n$ matrices. One can also work out what are $\mathbf{Pin}(2,0)$, $\mathbf{Pin}(1,1)$, and $\mathbf{Pin}(0,2)$; see Choquet-Bruhat [36], Chapter I, Section 7, page 26. Show that

$$\mathbf{Spin}(0,2) = \mathbf{Spin}(2,0) \approx \mathbf{U}(1),$$

and

$$\mathbf{Spin}(1,1) = \{a1 + be_1e_2 \mid a^2 - b^2 = 1\}.$$

Observe that $\mathbf{Spin}(1,1)$ is not connected.

More generally, it can be shown that $\mathrm{Cl}^0_{p,q}$ and $\mathrm{Cl}^0_{q,p}$ are isomorphic, from which it follows that $\mathbf{Spin}(p,q)$ and $\mathbf{Spin}(q,p)$ are isomorphic, but $\mathbf{Pin}(p,q)$ and $\mathbf{Pin}(q,p)$ are not isomorphic in general, and in particular, $\mathbf{Pin}(p,0)$ and $\mathbf{Pin}(0,p)$ are not isomorphic in general (see Choquet-Bruhat [36], Chapter I, Section 7). However, due to the "8-periodicity" of the Clifford algebras (to be discussed in the next section), it follows that $\mathrm{Cl}_{p,q}$ and $\mathrm{Cl}_{q,p}$ are isomorphic when $|p - q| = 0 \bmod 4$.

21.6 Periodicity of the Clifford Algebras $\mathrm{Cl}_{p,q}$

It turns out that the real algebras $\mathrm{Cl}_{p,q}$ can be build up as tensor products of the basic algebras $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. As pointed out by Lounesto (Section 23.16 [99]), the description of the real algebras $\mathrm{Cl}_{p,q}$ as matrix algebras and the 8-periodicity was first observed by Elie Cartan in 1908; see Cartan's article, *Nombres Complexes*, based on the original article in German by E. Study, in Molk [112], article I-5 (fasc. 3), pages 329-468. These algebras are defined in Section 36 under the name "'Systems of Clifford and Lipschitz," page 463-466. Of course, Cartan used a very different notation; see page 464 in the article cited above. These facts were rediscovered independently by Raoul Bott in the 1960's (see Raoul Bott's comments in Volume 2 of his Collected papers.).

We will use the notation $\mathbb{R}(n)$ (resp. $\mathbb{C}(n)$) for the algebra $M_n(\mathbb{R})$ of all $n \times n$ real matrices (resp. the algebra $M_n(\mathbb{C})$ of all $n \times n$ complex matrices). As mentioned in Example 21.3, it is not hard to show that

$$\begin{aligned} \mathrm{Cl}_{0,1} &= \mathbb{C} \qquad \mathrm{Cl}_{1,0} = \mathbb{R} \oplus \mathbb{R} \\ \mathrm{Cl}_{0,2} &= \mathbb{H} \qquad \mathrm{Cl}_{2,0} = \mathbb{R}(2) \end{aligned}$$

and

$$\mathrm{Cl}_{1,1} = \mathbb{R}(2).$$

The key to the classification is the following lemma:

Lemma 21.15 *We have the isomorphisms*

$$\begin{aligned} \mathrm{Cl}_{0,n+2} &\approx \mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2} \\ \mathrm{Cl}_{n+2,0} &\approx \mathrm{Cl}_{0,n} \otimes \mathrm{Cl}_{2,0} \\ \mathrm{Cl}_{p+1,q+1} &\approx \mathrm{Cl}_{p,q} \otimes \mathrm{Cl}_{1,1}, \end{aligned}$$

for all $n, p, q \geq 0$.

Proof. Let $\Phi_{0,n}(x) = -\|x\|^2$, where $\|x\|$ is the standard Euclidean norm on $\mathbb{R}^{n+2}$, and let $(e_1, \ldots, e_{n+2})$ be an orthonormal basis for $\mathbb{R}^{n+2}$ under the standard Euclidean inner product. We also let $(e'_1, \ldots, e'_n)$ be a set of generators for $\mathrm{Cl}_{n,0}$ and (e''_1, e''_2) be a set of generators for $\mathrm{Cl}_{0,2}$. We can define a linear map $f\colon \mathbb{R}^{n+2} \to \mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2}$ by its action on the basis $(e_1, \ldots, e_{n+2})$ as follows:

$$f(e_i) = \begin{cases} e'_i \otimes e''_1 e''_2 & \text{for } 1 \leq i \leq n \\ 1 \otimes e''_{i-n} & \text{for } n+1 \leq i \leq n+2. \end{cases}$$

Observe that for $1 \leq i, j \leq n$, we have

$$f(e_i)f(e_j) + f(e_j)f(e_i) = (e'_i e'_j + e'_j e'_i) \otimes (e''_1 e''_e)^2 = -2\delta_{ij} 1 \otimes 1,$$

since $e''_1 e''_2 = -e''_2 e''_1$, $(e''_1)^2 = -1$, and $(e''_2)^2 = -1$, and $e'_i e'_j = -e'_j e'_i$, for all $i \neq j$, and $(e'_i)^2 = 1$, for all i with $1 \leq i \leq n$. Also, for $n+1 \leq i, j \leq n+2$, we have

$$f(e_i)f(e_j) + f(e_j)f(e_i) = 1 \otimes (e''_{i-n} e''_{j-n} + e''_{j-n} e''_{i-n}) = -2\delta_{ij} 1 \otimes 1,$$

and

$$f(e_i)f(e_k) + f(e_k)f(e_i) = 2e'_i \otimes (e''_1 e''_2 e''_{n-k} + e''_{n-k} e''_1 e''_2) = 0,$$

for $1 \leq i, j \leq n$ and $n+1 \leq k \leq n+2$ (since $e''_{n-k} = e''_1$ or $e''_{n-k} = e''_2$). Thus, we have

$$f(x)^2 = -\|x\|^2 \cdot 1 \otimes 1 \quad \text{for all } x \in \mathbb{R}^{n+2},$$

and by the universal mapping property of $\mathrm{Cl}_{0,n+2}$, we get an algebra map

$$\widetilde{f}\colon \mathrm{Cl}_{0,n+2} \to \mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2}.$$

Since $\widetilde{f}$ maps onto a set of generators, it is surjective. However

$$\dim(\mathrm{Cl}_{0,n+2}) = 2^{n+2} = 2^n \cdot 2 = \dim(\mathrm{Cl}_{n,0})\dim(\mathrm{Cl}_{0,2}) = \dim(\mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2}),$$

and $\widetilde{f}$ is an isomorphism.

The proof of the second identity is analogous. For the third identity, we have

$$\Phi_{p,q}(x_1, \ldots, x_{p+q}) = x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2),$$

and let $(e_1, \ldots, e_{p+1}, \epsilon_1, \ldots, \epsilon_{q+1})$ be an orthogonal basis for $\mathbb{R}^{p+q+2}$ so that $\Phi_{p+1,q+1}(e_i) = +1$ and $\Phi_{p+1,q+1}(\epsilon_j) = -1$ for $i = 1, \ldots, p+1$ and $j = 1, \ldots, q+1$. Also, let $(e_1', \ldots, e_p', \epsilon_1', \ldots, \epsilon_q')$ be a set of generators for $\mathrm{Cl}_{p,q}$ and (e_1'', ϵ_1'') be a set of generators for $\mathrm{Cl}_{1,1}$. We define a linear map $f\colon \mathbb{R}^{p+q+2} \to \mathrm{Cl}_{p,q} \otimes \mathrm{Cl}_{1,1}$ by its action on the basis as follows:

$$f(e_i) = \begin{cases} e_i' \otimes e_1''\epsilon_1'' & \text{for } 1 \leq i \leq p \\ 1 \otimes e_1'' & \text{for } i = p+1, \end{cases}$$

and

$$f(\epsilon_j) = \begin{cases} \epsilon_j' \otimes e_1''\epsilon_1'' & \text{for } 1 \leq j \leq q \\ 1 \otimes \epsilon_1'' & \text{for } j = q+1. \end{cases}$$

We can check that

$$f(x)^2 = \Phi_{p+1,q+1}(x) \cdot 1 \otimes 1 \quad \text{for all } x \in \mathbb{R}^{p+q+2},$$

and we finish the proof as in the first case. □

To apply this lemma, we need some further isomorphisms among various matrix algebras.

Proposition 21.16 *The following isomorphisms hold:*

$$\begin{aligned} \mathbb{R}(m) \otimes \mathbb{R}(n) &\approx \mathbb{R}(mn) && \textit{for all } m, n \geq 0 \\ \mathbb{R}(n) \otimes_{\mathbb{R}} K &\approx K(n) && \textit{for } K = \mathbb{C} \textit{ or } K = \mathbb{H} \textit{ and all } n \geq 0 \\ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\approx \mathbb{C} \oplus \mathbb{C} \\ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} &\approx \mathbb{C}(2) \\ \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} &\approx \mathbb{R}(4). \end{aligned}$$

Proof. Details can be found in Lawson and Michelsohn [96]. The first two isomorphisms are quite obvious. The third isomorphism $\mathbb{C} \oplus \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$ is obtained by sending

$$(1, 0) \mapsto \frac{1}{2}(1 \otimes 1 + i \otimes i), \quad (0, 1) \mapsto \frac{1}{2}(1 \otimes 1 - i \otimes i).$$

The field $\mathbb{C}$ is isomorphic to the subring of $\mathbb{H}$ generated by $\mathbf{i}$. Thus, we can view $\mathbb{H}$ as a $\mathbb{C}$-vector space under left scalar multiplication. Consider the $\mathbb{R}$-bilinear map $\pi\colon \mathbb{C} \times \mathbb{H} \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{H})$ given by

$$\pi_{y,z}(x) = yx\overline{z},$$

where $y \in \mathbb{C}$ and $x, z \in \mathbb{H}$. Thus, we get an $\mathbb{R}$-linear map $\pi\colon \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{H})$. However, we have $\mathrm{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{H}) \approx \mathbb{C}(2)$. Furthermore, since

$$\pi_{y,z} \circ \pi_{y',z'} = \pi_{yy',zz'},$$

the map π is an algebra homomorphism

$$\pi\colon \mathbb{C}\times\mathbb{H}\to\mathbb{C}(2).$$

We can check on a basis that π is injective, and since

$$\dim_{\mathbb{R}}(\mathbb{C}\times\mathbb{H}) = \dim_{\mathbb{R}}(\mathbb{C}(2)) = 8,$$

the map π is an isomorphism. The last isomorphism is proved in a similar fashion. □

We now have the main periodicity theorem.

Theorem 21.17 *(Cartan/Bott) For all $n \geq 0$, we have the following isomorphisms:*

$$\begin{aligned}\mathrm{Cl}_{0,n+8} &\approx \mathrm{Cl}_{0,n}\otimes\mathrm{Cl}_{0,8}\\ \mathrm{Cl}_{n+8,0} &\approx \mathrm{Cl}_{n,0}\otimes\mathrm{Cl}_{8,0}.\end{aligned}$$

Furthermore,

$$\mathrm{Cl}_{0,8} = \mathrm{Cl}_{8,0} = \mathbb{R}(16).$$

Proof. By Lemma 21.15 we have the isomorphisms

$$\begin{aligned}\mathrm{Cl}_{0,n+2} &\approx \mathrm{Cl}_{n,0}\otimes\mathrm{Cl}_{0,2}\\ \mathrm{Cl}_{n+2,0} &\approx \mathrm{Cl}_{0,n}\otimes\mathrm{Cl}_{2,0},\end{aligned}$$

and thus,

$$\mathrm{Cl}_{0,n+8}\approx\mathrm{Cl}_{n+6,0}\otimes\mathrm{Cl}_{0,2}\approx\mathrm{Cl}_{0,n+4}\otimes\mathrm{Cl}_{2,0}\otimes\mathrm{Cl}_{0,2}\approx\cdots\approx\mathrm{Cl}_{0,n}\otimes\mathrm{Cl}_{2,0}\otimes\mathrm{Cl}_{0,2}\otimes\mathrm{Cl}_{2,0}\otimes\mathrm{Cl}_{0,2}.$$

Since $\mathrm{Cl}_{0,2}=\mathbb{H}$ and $\mathrm{Cl}_{2,0}=\mathbb{R}(2)$, by Proposition 21.16, we get

$$\mathrm{Cl}_{2,0}\otimes\mathrm{Cl}_{0,2}\otimes\mathrm{Cl}_{2,0}\otimes\mathrm{Cl}_{0,2}\approx\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{R}(2)\otimes\mathbb{R}(2)\approx\mathbb{R}(4)\otimes\mathbb{R}(4)\approx\mathbb{R}(16).$$

The second isomorphism is proved in a similar fashion. □

From all this, we can deduce the following table:

n	0	1	2	3	4	5	6	7	8
$\mathrm{Cl}_{0,n}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
$\mathrm{Cl}_{n,0}$	$\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$

A table of the Clifford groups $\mathrm{Cl}_{p,q}$ for $0\leq p,q\leq 7$ can be found in Kirillov [85], and for $0\leq p,q\leq 8$, in Lawson and Michelsohn [96] (but beware that their $\mathrm{Cl}_{p,q}$ is our $\mathrm{Cl}_{q,p}$). It can also be shown that

$$\begin{aligned}\mathrm{Cl}_{p+1,q} &\approx \mathrm{Cl}_{q+1,p}\\ \mathrm{Cl}_{p,q} &\approx \mathrm{Cl}_{p-4,q+4}\end{aligned}$$

with $p \geq 4$ in the second identity (see Lounesto [99], Chapter 16, Sections 16.3 and 16.4). Using the second identity, if $|p-q| = 4k$, it is easily shown by induction on k that $\mathrm{Cl}_{p,q} \approx \mathrm{Cl}_{q,p}$, as claimed in the previous section.

We also have the isomorphisms

$$\mathrm{Cl}_{p,q} \approx \mathrm{Cl}^0_{p,q+1},$$

frow which it follows that

$$\mathbf{Spin}(p,q) \approx \mathbf{Spin}(q,p)$$

(see Choquet-Bruhat [36], Chapter I, Sections 4 and 7). However, in general, $\mathbf{Pin}(p,q)$ and $\mathbf{Pin}(q,p)$ are not isomorphic. In fact, $\mathbf{Pin}(0,n)$ and $\mathbf{Pin}(n,0)$ are not isomorphic if $n \neq 4k$, with $k \in \mathbb{N}$ (see Choquet-Bruhat [36], Chapter I, Section 7, page 27).

21.7 The Complex Clifford Algebras $\mathrm{Cl}(n,\mathbb{C})$

One can also consider Clifford algebras over the complex field $\mathbb{C}$. In this case, it is well-known that every nondegenerate quadratic form can be expressed by

$$\Phi^{\mathbb{C}}_n(x_1,\ldots,x_n) = x_1^2 + \cdots + x_n^2$$

in some orthonormal basis. Also, it is easily shown that the complexification $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}_{p,q}$ of the real Clifford algebra $\mathrm{Cl}_{p,q}$ is isomorphic to $\mathrm{Cl}(\Phi^{\mathbb{C}}_n)$. Thus, all these complex algebras are isomorphic for $p+q=n$, and we denote them by $\mathrm{Cl}(n,\mathbb{C})$. Theorem 21.15 yields the following periodicity theorem:

Theorem 21.18 *The following isomorphisms hold:*

$$\mathrm{Cl}(n+2,\mathbb{C}) \approx \mathrm{Cl}(n,\mathbb{C}) \otimes_{\mathbb{C}} \mathrm{Cl}(2,\mathbb{C}),$$

with $\mathrm{Cl}(2,\mathbb{C}) = \mathbb{C}(2)$.

Proof. Since $\mathrm{Cl}(n,\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}_{0,n} = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}_{n,0}$, by Lemma 21.15, we have

$$\mathrm{Cl}(n+2,\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}_{0,n+2} \approx \mathbb{C} \otimes_{\mathbb{R}} (\mathrm{Cl}_{n,0} \otimes_{\mathbb{R}} \mathrm{Cl}_{0,2}) \approx (\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}_{n,0}) \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}_{0,2}).$$

However, $\mathrm{Cl}_{0,2} = \mathbb{H}$, $\mathrm{Cl}(n,\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}_{n,0}$, and $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \approx \mathbb{C}(2)$, so we get $\mathrm{Cl}(2,\mathbb{C}) = \mathbb{C}(2)$ and

$$\mathrm{Cl}(n+2,\mathbb{C}) \approx \mathrm{Cl}(n,\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}(2),$$

and the theorem is proved. □

As a corollary of Theorem 21.18, we obtain the fact that

$$\mathrm{Cl}(2k,\mathbb{C}) \approx \mathbb{C}(2^k) \quad \text{and} \quad \mathrm{Cl}(2k+1,\mathbb{C}) \approx \mathbb{C}(2^k) \oplus \mathbb{C}(2^k).$$

The table of the previous section can also be completed as follows:

n	0	1	2	3	4	5	6	7	8
$\mathrm{Cl}_{0,n}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
$\mathrm{Cl}_{n,0}$	$\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$\mathrm{Cl}(n,\mathbb{C})$	$\mathbb{C}$	$2\mathbb{C}$	$\mathbb{C}(2)$	$2\mathbb{C}(2)$	$\mathbb{C}(4)$	$2\mathbb{C}(4)$	$\mathbb{C}(8)$	$2\mathbb{C}(8)$	$\mathbb{C}(16)$.

where $2\mathbb{C}(k)$ is an abbrevation for $\mathbb{C}(k)\oplus\mathbb{C}(k)$.

21.8 The Groups Pin(p,q) and Spin(p,q) as double covers of O(p,q) and SO(p,q)

It turns out that the groups $\mathbf{Pin}(p,q)$ and $\mathbf{Spin}(p,q)$ have nice topological properties w.r.t. the groups $\mathbf{O}(p,q)$ and $\mathbf{SO}(p,q)$. To explain this, we review the definition of covering maps and covering spaces (for details, see Fulton [56], Chapter 11). Another interesting source is Chevalley [34], where is is proved that $\mathbf{Spin}(n)$ is a universal double cover of $\mathbf{SO}(n)$ for all $n\geq 3$.

Since $C_{p,q}$ is an algebra of dimension 2^{p+q}, it is a topological space as a vector space isomorphic to $V=\mathbb{R}^{2^{p+q}}$. Now, the group $C^*_{p,q}$ of units of $C_{p,q}$ is open in $C_{p,q}$, because $x\in C_{p,q}$ is a unit if the linear map $y\mapsto xy$ is an isomorphism, and $\mathbf{GL}(V)$ is open in $\mathrm{End}(V)$, the space of endomorphisms of V. Thus, $C^*_{p,q}$ is a Lie group, and since $\mathbf{Pin}(p,q)$ and $\mathbf{Spin}(p,q)$ are clearly closed subgroups of $C^*_{p,q}$, they are also Lie groups.

The definition below is analogous to the definition of a covering map given in Section 3.9 (Definition 3.33) except that now, we are only dealing with topological spaces and not manifolds.

Definition 21.7 Given two topological spaces X and Y, a *covering map* is a continuous surjective map, $p\colon Y\to X$, with the property that for every $x\in X$, there is some open subset, $U\subseteq X$, with $x\in U$, so that $p^{-1}(U)$ is the disjoint union of open subsets, $V_\alpha\subseteq Y$, and the restriction of p to each V_α is a homeomorphism onto U. We say that U is *evenly covered by* p. We also say that Y is a *covering space of* X. A covering map $p\colon Y\to X$ is called *trivial* if X itself is evenly covered by p (i.e., Y is the disjoint union of open subsets, Y_α, each homeomorphic to X), and *nontrivial*, otherwise. When each fiber, $p^{-1}(x)$, has the same finite cardinaly n for all $x\in X$, we say that p is an *n-covering* (or *n-sheeted covering*).

Note that a covering map, $p\colon Y\to X$, is not always trivial, but always *locally trivial* (i.e., for every $x\in X$, it is trivial in some open neighborhood of x). A covering is trivial iff Y is isomorphic to a product space of $X\times T$, where T is any set with the discrete topology. Also, if Y is connected, then the covering map is nontrivial.

Definition 21.8 An *isomorphism* φ between covering maps $p\colon Y\to X$ and $p'\colon Y'\to X$ is a homeomorphism, $\varphi\colon Y\to Y'$, so that $p=p'\circ\varphi$.

Typically, the space X is connected, in which case it can be shown that all the fibers $p^{-1}(x)$ have the same cardinality.

One of the most important properties of covering spaces is the path–lifting property, a property that we will use to show that $\mathbf{Spin}(n)$ is path-connected. The proposition below is the analog of Proposition 3.33 for topological spaces and continuous curves.

Proposition 21.19 *(Path lifting) Let $p\colon Y \to X$ be a covering map, and let $\gamma\colon [a,b] \to X$ be any continuous curve from $x_a = \gamma(a)$ to $x_b = \gamma(b)$ in X. If $y \in Y$ is any point so that $p(y) = x_a$, then there is a unique curve, $\widetilde{\gamma}\colon [a,b] \to Y$, so that $y = \widetilde{\gamma}(a)$ and*

$$p \circ \widetilde{\gamma}(t) = \gamma(t) \quad \text{for all } t \in [a,b].$$

Proof. See Fulton [57], Chapter 11, Lemma 11.6. □

Many important covering maps arise from the action of a group G on a space Y. If Y is a topological space, an *action (on the left) of a group G on Y* is a map $\alpha\colon G \times Y \to Y$ satisfying the following conditions, where, for simplicity of notation, we denote $\alpha(g,y)$ by $g \cdot y$:

(1) $g \cdot (h \cdot y) = (gh) \cdot y$, for all $g, h \in G$ and $y \in Y$;

(2) $1 \cdot y = y$, for all $\in Y$, where 1 is the identity of the group G;

(3) The map $y \mapsto g \cdot y$ is a homeomorphism of Y for every $g \in G$.

We define an equivalence relation on Y as follows: $x \equiv y$ iff $y = g \cdot x$ for some $g \in G$ (check that this is an equivalence relation). The equivalence class $G \cdot x = \{g \cdot x \mid g \in G\}$ of any $x \in Y$ is called the *orbit of* x. We obtain the quotient space Y/G and the projection map $p\colon Y \to Y/G$ sending every $y \in Y$ to its orbit. The space Y/G is given the quotient topology (a subset U of Y/G is open iff $p^{-1}(U)$ is open in Y).

Given a subset V of Y and any $g \in G$, we let

$$g \cdot V = \{g \cdot y \mid y \in V\}.$$

We say that G *acts evenly on* Y if for every $y \in Y$ there is an open subset V containing y so that $g \cdot V$ and $h \cdot V$ are disjoint for any two distinct elements $g, h \in G$.

The importance of the notion a group acting evenly is that such actions induce a covering map.

Proposition 21.20 *If G is a group acting evenly on a space Y, then the projection map, $p\colon Y \to Y/G$, is a covering map.*

Proof. See Fulton [57], Chapter 11, Lemma 11.17. □

The following proposition shows that $\mathbf{Pin}(p,q)$ and $\mathbf{Spin}(p,q)$ are nontrivial covering spaces unless $p = q = 1$.

Proposition 21.21 *For all $p, q \geq 0$, the groups* $\mathbf{Pin}(p, q)$ *and* $\mathbf{Spin}(p, q)$ *are double covers of* $\mathbf{O}(p, q)$ *and* $\mathbf{SO}(p, q)$*, respectively. Furthermore, they are nontrivial covers unless $p = q = 1$.*

Proof. We know that kernel of the homomorphism $\rho\colon \mathbf{Pin}(p, q) \to \mathbf{O}(p, q)$ is $\mathbb{Z}_2 = \{-1, 1\}$. If we let $\mathbb{Z}_2$ act on $\mathbf{Pin}(p, q)$ in the natural way, then $\mathbf{O}(p, q) \approx \mathbf{Pin}(p, q)/\mathbb{Z}_2$, and the reader can easily check that $\mathbb{Z}_2$ acts evenly. By Proposition 21.20, we get a double cover. The argument for $\rho\colon \mathbf{Spin}(p, q) \to \mathbf{SO}(p, q)$ is similar.

Let us now assume that $p \neq 1$ and $q \neq 1$. In order to prove that we have nontrivial covers, it is enough to show that -1 and 1 are connected by a path in $\mathbf{Pin}(p, q)$ (If we had $\mathbf{Pin}(p, q) = U_1 \cup U_2$ with U_1 and U_2 open, disjoint, and homeomorphic to $\mathbf{O}(p, q)$, then -1 and 1 would not be in the same U_i, and so, they would be in disjoint connected components. Thus, -1 and 1 can't be path–connected, and similarly with $\mathbf{Spin}(p, q)$ and $\mathbf{SO}(p, q)$.) Since $(p, q) \neq (1, 1)$, we can find two orthogonal vectors e_1 and e_2 so that $\Phi_{p,q}(e_1) = \Phi_{p,q}(e_2) = \pm 1$. Then,

$$\gamma(t) = \pm\cos(2t)\, 1 + \sin(2t)\, e_1e_2 = (\cos t\, e_1 + \sin t\, e_2)(\sin t\, e_2 - \cos t\, e_1),$$

for $0 \leq t \leq \pi$, defines a path in $\mathbf{Spin}(p, q)$, since

$$(\pm\cos(2t)\, 1 + \sin(2t)\, e_1e_2)^{-1} = \pm\cos(2t)\, 1 - \sin(2t)\, e_1e_2,$$

as desired. □

In particular, if $n \geq 2$, since the group $\mathbf{SO}(n)$ is path-connected, the group $\mathbf{Spin}(n)$ is also path-connected. Indeed, given any two points x_a and x_b in $\mathbf{Spin}(n)$, there is a path γ from $\rho(x_a)$ to $\rho(x_b)$ in $\mathbf{SO}(n)$ (where $\rho\colon \mathbf{Spin}(n) \to \mathbf{SO}(n)$ is the covering map). By Proposition 21.19, there is a path $\widetilde{\gamma}$ in $\mathbf{Spin}(n)$ with origin x_a and some origin $\widetilde{x_b}$ so that $\rho(\widetilde{x_b}) = \rho(x_b)$. However, $\rho^{-1}(\rho(x_b)) = \{-x_b, x_b\}$, and so, $\widetilde{x_b} = \pm x_b$. The argument used in the proof of Proposition 21.21 shows that x_b and $-x_b$ are path-connected, and so, there is a path from x_a to x_b, and $\mathbf{Spin}(n)$ is path-connected. In fact, for $n \geq 3$, it turns out that $\mathbf{Spin}(n)$ is simply connected. Such a covering space is called a *universal cover* (for instance, see Chevalley [34]).

This last fact requires more algebraic topology than we are willing to explain in detail, and we only sketch the proof. The notions of fibre bundle, fibration, and homotopy sequence associated with a fibration are needed in the proof. We refer the perseverant readers to Bott and Tu [19] (Chapter 1 and Chapter 3, Sections 16–17) or Rotman [128] (Chapter 11) for a detailed explanation of these concepts.

Recall that a topological space is *simply connected* if it is path connected and if $\pi_1(X) = (0)$, which means that every closed path in X is homotopic to a point. Since we just proved that $\mathbf{Spin}(n)$ is path connected for $n \geq 2$, we just need to prove that $\pi_1(\mathbf{Spin}(n)) = (0)$ for all $n \geq 3$. The following facts are needed to prove the above assertion:

(1) The sphere S^{n-1} is simply connected for all $n \geq 3$.

(2) The group $\mathbf{Spin}(3) \simeq \mathbf{SU}(2)$ is homeomorphic to S^3, and thus, $\mathbf{Spin}(3)$ is simply connected.

(3) The group $\mathbf{Spin}(n)$ acts on S^{n-1} in such a way that we have a fibre bundle with fibre $\mathbf{Spin}(n-1)$:

$$\mathbf{Spin}(n-1) \longrightarrow \mathbf{Spin}(n) \longrightarrow S^{n-1}.$$

Fact (1) is a standard proposition of algebraic topology and a proof can found in many books. A particularly elegant and yet simple argument consists in showing that any closed curve on S^{n-1} is homotopic to a curve that omits some point. First, it is easy to see that in $\mathbb{R}^n$, any closed curve is homotopic to a piecewise linear curve (a polygonal curve), and the radial projection of such a curve on S^{n-1} provides the desired curve. Then, we use the stereographic projection of S^{n-1} from any point omitted by that curve to get another closed curve in $\mathbb{R}^{n-1}$. Since $\mathbb{R}^{n-1}$ is simply connected, that curve is homotopic to a point, and so is its preimage curve on S^{n-1}. Another simple proof uses a special version of the Seifert—van Kampen's theorem (see Gramain [63]).

Fact (2) is easy to establish directly, using (1).

To prove (3), we let $\mathbf{Spin}(n)$ act on S^{n-1} via the standard action: $x \cdot v = xvx^{-1}$. Because $\mathbf{SO}(n)$ acts transitively on S^{n-1} and there is a surjection $\mathbf{Spin}(n) \longrightarrow \mathbf{SO}(n)$, the group $\mathbf{Spin}(n)$ also acts transitively on S^{n-1}. Now, we have to show that the stabilizer of any element of S^{n-1} is $\mathbf{Spin}(n-1)$. For example, we can do this for e_1. This amounts to some simple calculations taking into account the identities among basis elements. Details of this proof can be found in Mneimné and Testard [111], Chapter 4. It is still necessary to prove that $\mathbf{Spin}(n)$ is a fibre bundle over S^{n-1} with fibre $\mathbf{Spin}(n-1)$. For this, we use the following results whose proof can be found in Mneimné and Testard [111], Chapter 4:

Lemma 21.22 *Given any topological group G, if H is a closed subgroup of G and the projection $\pi\colon G \to G/H$ has a local section at every point of G/H, then*

$$H \longrightarrow G \longrightarrow G/H$$

is a fibre bundle with fibre H.

Lemma 21.22 implies the following key proposition:

Proposition 21.23 *Given any linear Lie group G, if H is a closed subgroup of G, then*

$$H \longrightarrow G \longrightarrow G/H$$

is a fibre bundle with fibre H.

Now, a fibre bundle is a fibration (as defined in Bott and Tu [19], Chapter 3, Section 16, or in Rotman [128], Chapter 11). For a proof of this fact, see Rotman [128], Chapter

11, or Mneimné and Testard [111], Chapter 4. So, there is a homotopy sequence associated with the fibration (Bott and Tu [19], Chapter 3, Section 17, or Rotman [128], Chapter 11, Theorem 11.48), and in particular, we have the exact sequence

$$\pi_1(\mathbf{Spin}(n-1)) \longrightarrow \pi_1(\mathbf{Spin}(n)) \longrightarrow \pi_1(S^{n-1}).$$

Since $\pi_1(S^{n-1}) = (0)$ for $n \geq 3$, we get a surjection

$$\pi_1(\mathbf{Spin}(n-1)) \longrightarrow \pi_1(\mathbf{Spin}(n)),$$

and so, by induction and (2), we get

$$\pi_1(\mathbf{Spin}(n)) \approx \pi_1(\mathbf{Spin}(3)) = (0),$$

proving that $\mathbf{Spin}(n)$ is simply connected for $n \geq 3$.

We can also show that $\pi_1(\mathbf{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$ for all $n \geq 3$. For this, we use Theorem 21.11 and Proposition 21.21, which imply that $\mathbf{Spin}(n)$ is a fibre bundle over $\mathbf{SO}(n)$ with fibre $\{-1, 1\}$, for $n \geq 2$:

$$\{-1, 1\} \longrightarrow \mathbf{Spin}(n) \longrightarrow \mathbf{SO}(n).$$

Again, the homotopy sequence of the fibration exists, and in particular, we get the exact sequence

$$\pi_1(\mathbf{Spin}(n)) \longrightarrow \pi_1(\mathbf{SO}(n)) \longrightarrow \pi_0(\{-1, +1\}) \longrightarrow \pi_0(\mathbf{SO}(n)).$$

Since $\pi_0(\{-1, +1\}) = \mathbb{Z}/2\mathbb{Z}$, $\pi_0(\mathbf{SO}(n)) = (0)$, and $\pi_1(\mathbf{Spin}(n)) = (0)$, when $n \geq 3$, we get the exact sequence

$$(0) \longrightarrow \pi_1(\mathbf{SO}(n)) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow (0),$$

and so, $\pi_1(\mathbf{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$. Therefore, $\mathbf{SO}(n)$ is not simply connected for $n \geq 3$.

Remark: Of course, we have been rather cavalier in our presentation. Given a topological space, X, the group $\pi_1(X)$ is the *fundamental group of* X, *i.e.*, the group of homotopy classes of closed paths in X (under composition of loops). But $\pi_0(X)$ is generally *not* a group! Instead, $\pi_0(X)$ is the set of path-connected components of X. However, when X is a Lie group, $\pi_0(X)$ is indeed a group. Also, we have to make sense of what it means for the sequence to be exact. All this can be made rigorous (see Bott and Tu [19], Chapter 3, Section 17, or Rotman [128], Chapter 11).